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VOL. 89.

THE ELEMENTS OF EUCLID,

WITH

ADDITIONAL PROPOSITIONS AND NOTES.

BY HENRY LAW, C.E.

PART II.

Price 1*s.*

LONDON :—JOHN WEALE.

THE

ELEMENTS OF EUCLID,

WITH

MANY ADDITIONAL PROPOSITIONS,

AND

EXPLANATORY NOTES;

TO WHICH IS PREFIXED AN

INTRODUCTORY ESSAY ON LOGIC.

BY

HENRY LAW,
CIVIL ENGINEER.

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London:
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1855.

183. c. 6.

PREFACE.

In presenting to the public the Second Volume of Euclid's Elements, the Author feels that some explanation is required for the interval which has elapsed since the publication of the former volume. He feels that it will be only necessary for him to state that the work has been written in the intervals snatched from his professional duties, which have occupied so considerable a share of his time as to leave him no choice between delaying the publication, or hurrying it forward in an imperfect form; and he felt that he would best promote the interests of the public, as well as his own reputation, by the adoption of the former alternative.

H. L.

15, Essex Street, Strand.
15th May, 1855.

THE

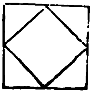
ELEMENTS OF EUCLID.

BOOK IV.

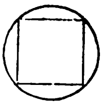
DEFINITIONS.

1. A rectilineal figure is said to be *inscribed* in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed each upon each.

2. In like manner, a figure is said to be *circumscribed* about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is circumscribed, each through each.

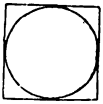


3. A Rectilineal figure is said to be *inscribed* in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

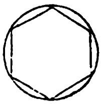


4. A Rectilineal figure is said to be *circumscribed* about a circle, when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner, a circle is said to be *inscribed* in a rectilinear figure, when the circumference of the circle touches each side of the figure.



6. A circle is said to be *circumscribed* about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is circumscribed.



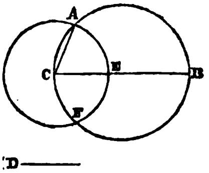
7. A straight line is said to be placed in a circle, when its extremities are in the circumference of the circle.

Scholium. A *regular* polygon is one which has all its sides or angles equal; in the first case it is said to be *equilateral*, and in the second, *equiangular*. Polygons further receive particular names, according to the number of sides which they possess, thus:—

A *Trigon*	is a polygon with	3	sides.
Tetragon	"	4	"
Pentagon	"	5	"
Hexagon	"	6	"
Heptagon	"	7	"
Octagon	"	8	"
Nonagon	"	9	"
Decagon	"	10	"
Undecagon	"	11	"
Duodecagon	"	12	"

PROPOSITION I.

Problem.—In a given *circle* (ABC) to inscribe a straight line, equal to a given *straight line* (D), which is not greater than the diameter of the circle.



Solution. *Draw a diameter* BC *of the circle; and if this be equal to the given line* D, *the problem is solved; but if it is not, take in it the segment* CE *equal to* D (*a*), *and from* C *as a center, with the radius* CE, *describe the circle* AEF, *and join* CA.

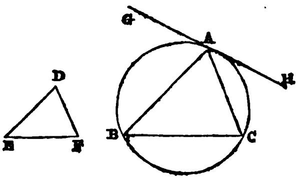
Demonstration. Because C is the center of the circle AEF, CA is equal to CE (*b*); but D is equal to CE (*c*), therefore D *is equal to* CA (*d*).

(*a*) I. 3.
(*b*) I. Def. 15.
(*c*) Solution.
(*d*) Ax. 1.

Scholium. It should be observed that in the enunciation of the above proposition, the word "given" is used in a different sense as applied to the circle and to the straight line, the former being given both in *position* and *magnitude*, while the latter is given only in *magnitude*.

PROPOSITION II.

Problem.—In a given *circle* (ABC) to inscribe a triangle equiangular to a given *triangle* (DEF).



Solution. *Draw the straight line* GAH *touching the circumference of the circle in the point* A (*a*), *and at the point* A *in the straight line* AH, *and on the same side of it with the circle form the angle* HAC *equal to the angle* E (*b*), *and at the same point in the straight line* AG, *and on the same side of it, form the angle* GAB *equal to the angle* F (*b*); *and since* AC *and* AB *are drawn from* A *between the tangent and the circumference, they must cut the circumference* (*c*); *let them do so respectively in the points* C *and* B; *then join* B *and* C.

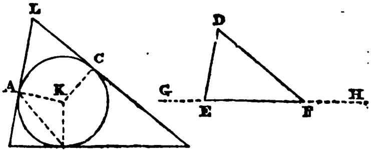
(*a*) III. 17.
(*b*) I. 23.
(*c*) III. 16.
(*d*) III. 32.
(*e*) Solution.
(*f*) Ax. 1.
(*g*) I. 32 B, cor. 3.

Demonstration. Because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal to the angle B in the alternate segment of the circle (*d*); but the angle HAC is equal to the angle E (*e*); therefore the angle B is equal to the angle E (*f*); and in the same manner it may be shown that the angle C is equal to the angle F; therefore the remaining angle D is equal to the angle BAC (*g*), and therefore *the triangle* ABC, *inscribed in the given circle, is equiangular to the given triangle* DEF.

Scholium. In the solution of this problem, Euclid has omitted to state that the lines AC and AB must be drawn on the same side of the tangent as the circle, and he has assumed that these lines will cut the circumference, without showing the reason of their doing so.

PROPOSITION III.

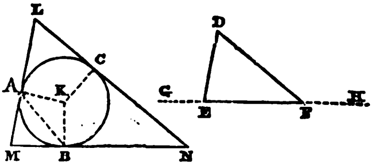
Problem.—About a given *circle* (ABC) to circumscribe a triangle equiangular to a given *triangle* (DEF).



Solution. *Produce* EF *both ways to* G *and* H; *find the center* K *of the circle* ABC (*a*), *and from it draw the straight line* KB; *at the point* K *in the straight line* KB *form the angle* BKA *equal to the angle* DEG (*b*), *and from the same point, and on the other side of the same*

(*a*) III. 1.
(*b*) I. 23.

straight line, form the angle CKB *equal to the anyle* DFH (*b*); *through the points* A, B, *and* C *draw the straight lines* ML, MN, *and* NL, *touching the circle* ABC (*c*), *then shall they meet in the points* M, N, *and* L, *and form the triangle required.*



(*b*) I. 23.
(*c*) III. 17.
(*d*) III. 18.
(*e*) Theor. attached to I. 29.
(*f*) I. 32 B, cor. 7.
(*g*) I. 13.
(*h*) Solution.
(*i*) Ax. 3.
(*k*) I. 32 B, cor. 3.

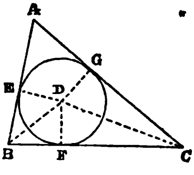
Demonstration. Join A and B, then because KAM and KBM are right angles (*d*), the angles BAM and ABM are less than two right angles, and therefore the lines AM and BM must meet one another, if produced far enough (*e*), let them meet in M, and in a similar manner it may be shown that AL and CL must meet in some point L, and that BN and CN must meet in some point N. Because the four angles of the quadrilateral figure AKBM are together equal to four right angles (*f*), and the angles KAM and KBM are right angles (*d*), the other two M and AKB are together equal to two right angles; but the angles DEG and DEF are together equal to two right angles (*g*), therefore the angles AKB and M are together equal to the angles DEG and DEF; but AKB and DEG are equal (*h*), and therefore M and DEF are equal (*i*). In the same manner it may be shown that the angle N is equal to DFE; therefore the remaining angle L is equal to the remaining angle D (*k*); and therefore *the triangle* LMN *circumscribed about the circle* ABC *is equiangular to the given triangle.*

Scholium. The demonstration of this proposition has been somewhat altered from that of Euclid, who omits to prove that the lines MN, LM, and LN must necessarily meet when produced.

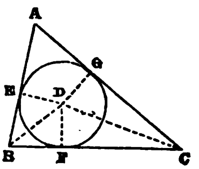
PROPOSITION IV.

Problem.—To inscribe a circle in a given *triangle* (ABC).

Solution. *Bisect any two angles* ABC *and* ACB (*a*), *by the straight lines* BD *and* CD, *then because the angles* ABC *and* ACB *are together less than two right angles* (*b*), *much more are* DBC *and* DCB *together less than two right angles; therefore* DB *and* DC *will meet, if produced far enough* (*c*), *let them meet in* D. *Then from* D *draw* DF



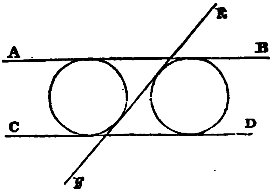
(*a*) I. 9.
(*b*) I. 17.
(*c*) Theor. attached to I. 29.



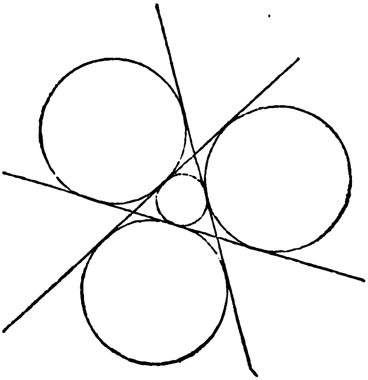
perpendicular to any side BC (*d*), *and from* D *as a center, and with the distance* DF *describe a circle* EFG *which shall be inscribed in the given triangle.*

DEMONSTRATION. From D draw DE and DG perpendicular to AB and AC. Then the angle ABC being bisected by DB (*e*), the angles EBD and FBD are equal, and the angles DEB and DFB being both right angles (*f*) are also equal, therefore the triangles EBD and FBD have two angles of the one respectively, equal to two angles of the other, and the side BD common to both, and therefore their other sides ED and FD are equal (*g*). In the same manner it may be shown that GD is equal to FD; therefore the three lines ED, FD, and GD are equal (*h*), and therefore the circle described from the center D, with the radius DF, passes through the points E and G, and because the angles at F, E, and G are right angles, the lines BC, AB, and AC are tangents to the circle (*i*); therefore *the circle* FEG *is inscribed in the given triangle* (*k*).

(*d*) I. 12.
(*e*) Solution.
(*f*) III. 18.
(*g*) I. 26.
(*h*) Ax. 1.
(*i*) III. 16.
(*k*) III. Def.



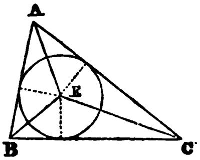
SCHOLIUM. The above proposition is only a particular case of the more general problem, "To describe a circle touching three given straight lines." 1°. If the three given lines are parallel to each other; or 2°. If they intersect at the same point the problem is impossible; 3°. If two of the lines, AB and CD, are parallel, and the third, EF, intersect them, it is possible to describe two equal circles, each fulfilling the conditions of the problem, one on either side of the line EF; 4°. If the three given lines intersect so as to form a triangle, four circles may be described, touching them, one inscribed as above, and the other three touching each of the sides of the triangle externally, and the other sides produced.



COROLLARY 1. The straight lines bisecting the three angles of a triangle meet in the center of the inscribed circle.

COROLLARY 2. A triangle is equal in area to the rectangle under the radius of the inscribed circle, and half the sum of the three sides or perimeter of the triangle.

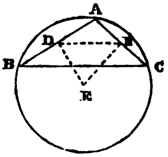
For the area of the whole triangle ABC is equal to the areas of the three triangles AEB, BEC, and AEC, and the area of each of these triangles is respectively equal to that of the rectangle, under the radius and half the sides AB, BC, and AC.

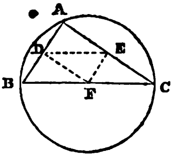


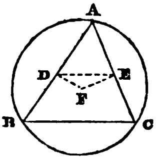
PROPOSITION V.

Problem.—To circumscribe a circle about a given *triangle* (ABC).

Solution. The three angular points, A, B, and C, of the triangle, are not in the same straight line, therefore a circle may be described passing through them in the manner demonstrated in the theorem attached to III. 1.







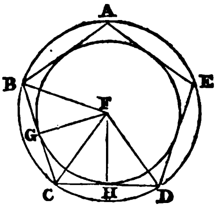
Scholia. 1. This proposition has been anticipated by the theorem above-mentioned.

2. If the center F fall within the triangle all its angles are acute, for each of them is in a segment greater than a semicircle. If the center be in any side of the triangle, the angle opposite that side is a right angle, because it is in a semicircle. And if the center fall without the triangle, the angle opposite to the side which is nearest to the center is an obtuse angle, because it is in a segment greater than a semicircle.

3. The two following propositions are here introduced, in order to simplify the demonstration of several of the subsequent problems.

PROPOSITION V. A.

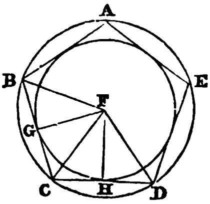
Theorem.—*If a rectilineal figure* (ABCDE) *be equilateral and equiangular,* [1] it may have one circle circumscribed about it, [2] and another inscribed in it; [3] and the same point is the center of both circles.



Construction. *Bisect the angles* BCD *and* CDE (a), *by the straight lines* CF *and* DF, *then because the angles* FCD *and* FDC

(a) I. 9.

are together less than two right angles, therefore CF *and* DF *will meet, if produced far enough* (*b*), *let them meet in* F. *Join* BF, *and from* F *draw* GF *and* HF *respectively perpendicular to* BC *and* CD (*c*).



DEMONSTRATION. [1.] In the triangle FCD, the angles FCD and FDC are equal, being the halves of equal angles, therefore the opposite sides CF and DF are equal (*d*). Also in the triangles FBC and FDC, the side BC is equal to CD (*e*), the side CF common to both, and the angle FCB equal to FCD (*f*), therefore the side BF is equal to DF (*g*). In the same manner it may be shown that the straight lines from F to the other angles A and E are equal to DF, and therefore *a circle described from* F *as a center, with the radius* DF, *will pass through all the angular points, and circumscribe the rectilineal figure* ABCDE.

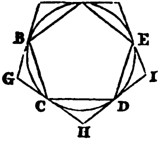
(*b*) Theor. attached to I. 29.
(*c*) I. 12.
(*d*) I. 6.
(*e*) Hypoth.
(*f*) Const.
(*g*) I. 4.
(*h*) III. 14.

[2.] Because BC and CD are equal chords of the circumscribed circle, they are equally distant from its center (*h*), that is, GF is equal to HF; in the same manner it may be shown that the perpendiculars drawn from F to the other sides AB, AE, and DE are all equal to HF, and therefore *that a circle described from* F *as a center, with the radius* HF, *will touch all the sides of the rectilineal figure* ABCDE, *and be inscribed in it.*

[3.] It is evident that *the same point* F *is the center of both the circumscribed and inscribed circles.*

PROPOSITION V. B.

THEOREM.—*If any equilateral and equiangular rectilineal figure* (ABCDE) *be inscribed in a circle*, tangents to the circle, drawn through the angular points, will form an equilateral and equiangular figure of the same number of sides, circumscribed about the circle.

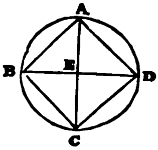


DEMONSTRATION. Because the chords AB and BC, &c., are equal, their arcs are also equal (*a*), and the angles FAB, FBA, GBC, GCB, &c., at the circumference standing on these arcs are also equal (*b*). Therefore in the triangles ABF, BCG, GHD, &c., the sides AF and BF, BG, GC, CH, &c., are all equal, and the angles F, G, H, &c., are also all equal (*c*), therefore *the rectilineal figure*, FGHIK, *circumscribed about the circle, is equilateral and equiangular.*

(*a*) III. 28.
(*b*) III. 27.
(*c*) I. 6.

PROPOSITION VI.

PROBLEM.—To inscribe a square in a given *circle* (ABCD).



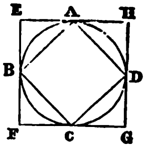
SOLUTION. *Draw the diameters* AC, BD, *at right angles to each other; and join* AB, BC, CD, *and* DA, *then* ABCD *is the square required.*

DEMONSTRATION. Because in the triangles BEA and AED, BE and ED are equal, AE common to both, and at right angles to BD, the base AB is equal to AD (*a*); and in the same manner it may be shown that each of the other sides, DC and BC are equal to AB, and therefore *that the quadrilateral figure* ABCD *is equilateral.* But the straight line BD being a diameter, ABD is a semicircle, and therefore *the angle* BAD *is a right angle* (*b*), *and the quadrilateral figure* ABCD *is a square.*

(*a*) I. 4.
(*b*) III. 31.

PROPOSITION VII.

PROBLEM.—To circumscribe a square about a given *circle* (ABCD).



SOLUTION. *Inscribe a square in the circle* ABCD (*a*), *and through its angular points,* A, B, C, *and* D, *draw tangents* EH, EF, FG, *and* GH (*b*), *then* EFGH *is the square required.*

DEMONSTRATION. Because the tangents, EH, EF, FG, and GH are drawn through the angular points of a square inscribed in a circle, *therefore they form a square* EFGH, *circumscribed about the same circle* (*c*).

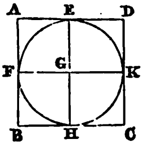
(*a*) IV. 6.
(*b*) III. 17.
(*c*) IV. 5 B.

COROLLARY. *If a square is circumscribed about a circle*, it is evidently equal in area to twice the square inscribed in the circle.

SCHOLIUM. It is evident that a square is the only right-angled parallelogram which can be circumscribed about a circle, but that either a square or rectangle may be inscribed in it.

PROPOSITION VIII.

Problem.—To inscribe a circle in a given *square* (ABCD).



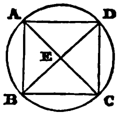
Solution. *Bisect each of the sides* AB, AD, *in the points* F *and* E (*a*); *through* F *draw* EK *parallel to* AD, *and through* E *draw* EH *parallel to* AB (*b*); *the circle* EFHK, *described from the center* G, *with the radius* EG, *is inscribed in the given square.*

Demonstration. Because AE, ED, AF, and FB are halves of equals, they are all equal to each other (*c*); and because AG, EK, FH, and GC are parallelograms (*d*), their opposite sides are equal (*e*); therefore, EG, FG, HG, and GK are all equal, and the circle described from the center G with the radius EG will pass through the points F, H, and K, and because the angles at E, F, H, and K, are right angles (*d*), the sides of the square are tangents to the circle EFHK (*f*); *which is therefore inscribed in the given square.*

(*a*) I. 10.
(*b*) I. 31.
(*c*) Ax. 7.
(*d*) Solution.
(*e*) I. 34.
(*f*) III. 16.

PROPOSITION IX.

Problem.—To circumscribe a circle about a given *square* (ABCD).



Solution. *Join* AC *and* BD, *cutting each other in* E; *the circle described from* E *as a center with the radius* AE *will circumscribe the given square.*

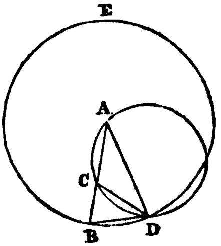
Demonstration. Because the triangle ABD is isosceles, and the angle A a right angle, therefore each of the angles ADB and ABD is half a right angle (*a*), and in the same manner it may be shown that each of the angles into which the angles of the square are divided by the diagonals is half a right angle; and, therefore, that they are all equal. Then in the triangle, AEB, as the angles A and B are equal, the opposite sides, BE and AE, are equal (*b*); and in the same manner it may be shown that CE and DE are equal to BE and AE, therefore the four lines, AE, BE, CE, and DE, are equal, and therefore the circle described from the center E with the radius AE passes through the angular points, A, B, C, and D, *and is circumscribed about the given square.*

(*a*) I. 32 B, cor. 2.
(*b*) I. 6.

PROPOSITION X.

Problem.—To construct an isosceles triangle, in which each of the angles at the base shall be double of the angle opposite to the same.

Solution. *Take any straight line* AB, *and divide it in* C, *so that the rectangle under* AB *and* BC *may be equal in area to the square on* AC (*a*); *construct the triangle,* ABD, *having* AD *equal to* AB, *and* DB *to* AC (*b*), *and it will be the triangle required. Join* CD, *and about the triangle* ACD *circumscribe the circle* ADC (*c*).



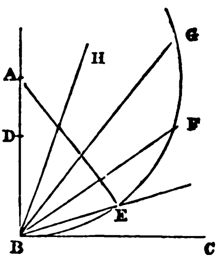
Demonstration. Because the rectangle under AB and BC is equal in area to the square on AC (*d*), or to the square on BD (*d*), the line BD is a tangent to the circle ADC (*e*), and therefore the angle BDC is equal to the angle A in the alternate segment (*f*); add to both the angle CDA, and BDA is equal to the sum of the angles CDA and A; but because the sides AB and AD are equal, therefore the opposite angles B and BDA are equal (*g*), and the angle B is equal to the sum of the angles CDA and A; but the external angle BCD is equal to the sum of the angles CDA and A (*h*), therefore the angles B and BCD are equal, and the sides BD and CD are equal (*i*); but BD and CA are equal (*d*), therefore CD and CA are equal, and therefore the angles A and CDA are equal (*g*), but BDA is equal to the sum of the angles A and CDA, *therefore the angles* BDA *and* B *are each double of the angle* A.

(*a*) II. 11.
(*b*) I. 22.
(*c*) IV. 5.
(*d*) Solution.
(*e*) III. 37.
(*f*) III. 82.
(*g*) I. 5.
(*h*) I. 32 A.
(*i*) I. 6.

Corollary 1. The triangle BDC is also isosceles, and has each of the angles at its base, B and BCD, double of the vertical angle BDC.

Corollary 2. The triangle ACD is isosceles, and each of the angles at its base, A and ADC, are one-third of the vertical angle ACD.

Corollary 3. Problem. To divide a given *right angle* (ABC) into five equal parts.



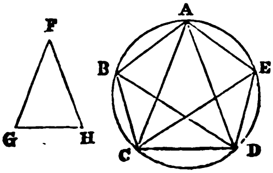
Solution. *In* AB *take any point* A, *and divide* AB *in* D, *so that the rectangle under* AB *and* AD *shall equal in area the square on* DB (*a*). *From* A *as a center, with the radius* AB *describe a circle, and in it place* BE *equal to* DB (*b*); *the angle* EBC *is a fifth of the right angle.*

Demonstration. Because in the triangle ABE the angles ABE and AEB are each equal to the double of the angle A (*c*), and the three angles ABE, AEB, and A, are together equal to two right angles (*d*), therefore the angle A equals two-fifths of a right, and the angle ABE equals four-fifths of a right angle, and therefore the remaining angle EBC equals one-fifth of the right angle ABC, and if the angle ABE be divided into four equal parts (*e*) by the lines BF, BG, and BH, the whole right angle ABC will be divided into five equal parts.

(*a*) II. 11.
(*b*) I. 2.
(*c*) IV. 10.
(*d*) I. 32 B.
(*e*) I. 9.

PROPOSITION XI.

Problem.—To inscribe an equilateral and equiangular pentagon in a given *circle* (ABCDE).



Solution. *Construct an isosceles triangle,* FGH, *having each of the angles at* GH, *double of the angle at* F (*a*); *and in the circle* ABCDE *inscribe the triangle* ACD *equiangular to the triangle* FGH (*b*). *Bisect the angles at the base,* ADC *and* ACD, *by the straight lines* BD *and* EC (*c*), *and join* CB, BA, AE, *and* ED; *then* ABCDE *is the pentagon required.*

Demonstration. Because each of the angles ACD and ADC is double of CAD, and is bisected (*d*), the five angles, DAC, ACE, ECD, CDB, and BDA, are equal to one another; therefore the arcs upon which they stand are equal (*e*), and therefore the straight lines, DC, AE, ED, CB, and BA, which subtend those arcs, are also equal (*f*); and therefore *the pentagon* ABCDE *is equilateral.* And because the arcs AB and DE are equal, if the arc BCD be added to both, the arc ABCD is equal to the arc BCDE (*g*), and therefore the angles AED and BAE, standing upon them, are equal (*e*); and in the same manner it may be shown that all the other angles are equal to one another, and therefore *that the pentagon* ABCDE *is also equiangular.*

(*a*) IV. 10.
(*b*) IV. 2.
(*c*) I. 9.
(*d*) Solution.
(*e*) III. 26.
(*f*) III. 29.
(*g*) Ax. 2.

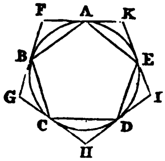
Corollary. Hence it is evident that every equiangular figure inscribed in a circle is equilateral, and that every equilateral figure is equiangular.

Scholium. In the above proposition the pentagon is inscribed in the circle by the aid of an isosceles triangle, the angles at whose base are each double its vertical angle; and in like manner any other equilateral figure of any number of sides may be inscribed in a circle, by the aid of an isosceles triangle, in which each of the angles at its base is to its vertical angle as half the number of its sides minus half, is to unity; thus a square may be inscribed by the aid of an isosceles triangle having the ratio between each of the angles at its base and its vertical angle as ($\frac{4}{2} - \frac{1}{2} =$) $1\frac{1}{2} : 1$; a pentagon, as ($\frac{5}{2} - \frac{1}{2} =$) $2 : 1$; a hexagon, as ($\frac{6}{2} - \frac{1}{2} =$) $2\frac{1}{2} : 1$; and so on.

PROPOSITION XII.

Problem.—To circumscribe an equilateral and equiangular pentagon about a given circle (ABCDE).

Solution. Inscribe within the given circle the equilateral and equiangular pentagon ABCDE (*a*), then through the angular points of the same, A, B, C, &c., draw tangents KF, FG, GH, &c., to the given circle (*b*), and they will form an equilateral and equiangular pentagon, FGHIK, circumscribing the given circle (*c*).

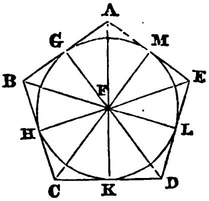


(*a*) IV. 11.
(*b*) III. 17.
(*c*) IV. 5 B.

PROPOSITION XIII.

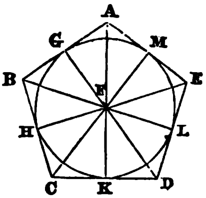
Problem.—To inscribe a circle in a given *equilateral and equiangular pentagon* (ABCDE).

Solution. *Bisect any two adjacent angles* A *and* B *by the straight lines* AF *and* BF (*a*), *and from their point of intersection* F *draw* FG *perpendicular to* AB (*b*); *from the center* F *with the radius* FG *describe a circle, and it will be inscribed in the given pentagon. Draw* FC, FD, FE, *and from* F *let fall the perpendiculars* FH, FK, FL, *and* FM (*b*).



Demonstration. In the triangles ABF and AEF the sides AB and AE are equal (*c*), AF common to both, and FAB and FAE are equal (*d*), therefore the angles ABF and AEF are equal (*e*); but the angles ABC and AED are also equal (*c*), therefore, since

(*a*) I. 9.
(*b*) I. 12.
(*c*) Hypo.
(*d*) Solution.
(*e*) I. 4.



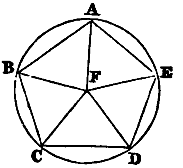
ABF is half of ABC (*d*), AEF is half of AED; and in the same manner it may be shown that the other angles BCD and CDE are bisected by the lines FC and FD. Therefore in the triangles FBH and FBG the angles FBH and FBG are equal, the angles BHF and BGF are right angles (*d*), and the side FB common, therefore the sides FH and FG are equal (*f*); and in the same manner it may be shown that all the perpendiculars FH, FK, FL, &c., are equal, therefore *the circle described from* F *as a center with the radius* FG *will pass through the points* H, K, L, *and* M, *and the sides of the given pentagon are tangents to it because the angles at those points are right angles* (*g*).

(*d*) Solution.
(*f*) I. 26.
(*g*) III. 16.

SCHOLIUM. This problem is only a particular case of the more general proposition given at p. 6.

PROPOSITION XIV.

PROBLEM.—To circumscribe a circle about a given *equilateral and equiangular pentagon* (ABCDE).

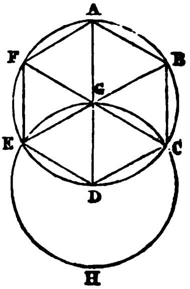


SOLUTION. *Bisect the angles* A *and* E *by the straight lines* AF *and* EF (*a*); *from the point of intersection* F *as a center with the radius* AF, *describe a circle* ABCDE *which shall circumscribe the given pentagon. Draw the straight lines* BF, CF, *and* DF.

DEMONSTRATION. It may be shown in the same manner as in the preceding proposition that the angles of the pentagon are bisected by the straight lines drawn from F. Therefore in the triangle AFE the angle EAF is equal to AEF, and therefore the side AF is equal to FE (*b*); and in the same manner it may be shown that all the lines AF, BF, CF, DF, and EF are equal, and therefore *the circle described from* F *as a center with the radius* AF *will pass through the points* B, C, D, E, *and circumscribe the pentagon.*

(*a*) I. 9.
(*b*) I. 6.

PROPOSITION XV.



Problem.—To inscribe an equilateral and equiangular hexagon in a given circle (ABCDEF).

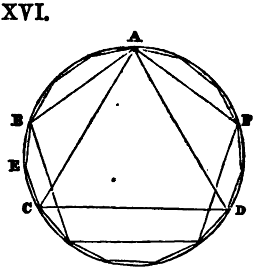
Solution. *Find the center* G *of the given circle* (*a*), *and through it draw the diameter* AD. *From* D *as a center with the radius* DG *describe a circle* GEHC, *join* EG *and* GC, *and produce them to the points* B *and* F. *Join* AF, FE, ED, DC, CB, BA, *with straight lines, and they will form an equilateral and equiangular hexagon.*

Demonstration. The straight lines GD and DC, being radii of the same circle, are equal (*b*), and for the same reason DG and GC are equal, therefore the triangle DGC is equilateral, and the angle CGD is the third part of two right angles (*c*); and in the same manner it may be shown that the angle EGD is also the third part of two right angles. And because the straight line GC makes with EB the adjacent angles EGC, CGB, equal to two right angles (*d*), the remaining angle CGB is the third part of two right angles, and the three angles EGD, DGC, and CGB are equal to one another; and to these the vertical opposite angles BGA, AGF, and FGE are also equal (*e*); therefore the six angles at the center G are equal, and the arcs on which they stand are equal (*f*), and also the lines subtending those arcs are equal (*g*), and *therefore the hexagon* ABCDEF *is equilateral, and also, since it is inscribed in a circle, equiangular* (*h*).

(*a*) III. 1.
(*b*) I. Def. 15.
(*c*) I. 32 B, cor. 5.
(*d*) I. 13.
(*e*) I. 15.
(*f*) III. 26.
(*g*) III. 29.
(*h*) IV. 11. cor.

Corollary. It is evident that the side of the hexagon is equal to the radius of the circumscribing circle.

PROPOSITION XVI.



Problem.—To inscribe an equilateral and equiangular quindecagon in a given *circle* (ABCD).

Solution. *Let* AC *be the side of an equilateral triangle inscribed in the circle* (*a*), *and* AB *the side of an equilateral pentagon inscribed in the same* (*b*); *bisect the arc* BC *in* E (*c*), *join* BE *and* EC, *and in the given circle place chords equal to* BE, *and they will form an equilateral and equiangular quindecagon inscribed in it.*

(*a*) IV. 2.
(*b*) IV. 11.
(*c*) III. 30.

DEMONSTRATION. For if the whole circumference of the given circle be divided into fifteen equal parts, the arc AC, because it is the third part of the whole circumference, contains five of these parts; in like manner the arc AB contains three of them, therefore the arc BC contains two, and therefore the arc BE is the fifteenth part of the whole circumference, *and* BE *is the side of the required equilateral and equiangular quindecagon.*

SCHOLIUM. The only regular polygons which the Greek Geometers could incribe geometrically in the circle were the trigon, or equilateral triangle, the tetragon, or square, the pentagon, the hexagon, and any others, such as the quindecagon, derived from them. M. Gause, however, in his *Disquisitiones Arithmeticæ*, has shown that a regular polygon of $2^n + 1$ sides is always capable of being inscribed geometrically in a circle, when $2^n + 1$ is a prime number.

THE

ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

1. A LESS magnitude is said to be a *part* of a greater magnitude, when the less measures the greater, that is, when the less is contained a certain number of times exactly in the greater.

SCHOLIUM. In ordinary use the word "part" means "any portion whatever," but its geometrical sense in the above definition, and wherever subsequently employed, is that of an *aliquot* part or *submultiple*. It has already been explained in the scholium to the first proposition of the second book, that one magnitude is said to *measure* another when it is exactly contained in it any number of times without any remainder. The lesser magnitude is then said to be a *part* or *submultiple* of the greater, while the greater is said to be a *multiple* of the less.

In the four preceding books magnitudes have been compared simply as to their *equality* or *inequality*, but in the latter case no attempt has been made to determine how great or how small that inequality might be. The object, however, of the fifth book is to compare unequal magnitudes, and to determine with greater exactness their *relative value*. Now there are two ways in which two unequal magnitudes or quantities might be compared, namely,—1°, by subtracting the lesser from the greater, and so ascertaining how much one exceeded the other; thus if one line were represented by 50 and the other by 40, their difference thus estimated would be 10; this method, however, would fail to convey any idea of their *relative values*;—2°, by ascertaining how often the greater contained the less, or, in other words, what multiple the greater was of the less; this latter method is the one employed by Euclid in the fifth book, and by it we are enabled to ascertain their *relative value*.

2. A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

SCHOLIUM. It is necessary to observe the distinction between the expressions "measures" and "is contained in;" for example, 3 *measures* 15, being contained in it exactly 5 times without any remainder, but 3 does not measure 13, although it *is contained in it* 4 times, because there is a remainder of 1 over. It has already been explained, in the scholium to

II. 1, that when two magnitudes are *multiples* of the same magnitude, or, in other words, when they may both be *measured* by the same magnitude, they are said to be *commensurable*, but that when no magnitude could be found by which both the given magnitudes could be measured, they were said to be *incommensurable*, as in the case of the side and diagonal of a square.

3. *Ratio* is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.

SCHOLIUM. This definition has been as severely criticised as perhaps any other portion of the Elements; but it should be borne in mind that no subsequent conclusions are deduced from, or made to depend upon it, but that Euclid doubtless introduced it as a mere explanation of the sense in which the word "ratio" was to be afterwards employed. There is, however, a defect in the definition, inasmuch as it is not stated in what way the comparison of the two magnitudes is to be made, for we have already mentioned that two modes of comparison may be adopted, namely, either by finding the excess of one magnitude above the other, or by ascertaining what multiple one is of the other. In the following definition given by Wood in his Algebra, this objection is removed:—"*Ratio* is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part or parts, one is of the other."

In order that two magnitudes may be capable of comparison so as to determine their ratio, it is essential that they should be of the "same kind," that is to say, two lines, two angles, two surfaces, or two solids; or, as is expressed in the next definition, they must be such that "the less may be multiplied so as to exceed the greater."

It cannot be too strongly impressed on the learner that the *ratio* of two quantities is entirely irrespective of their *actual* magnitude, but is determined solely by their *relative* magnitude; so that if any ratio has been found to exist between any two quantities, that ratio will remain unaltered, although the original quantities may be both doubled or both halved, or, in fact, multiplied or divided by any other quantity, or submitted to any other operation.

The two quantities between which the ratio exists, are called the *terms of the ratio;* the first being named the *antecedent* and the second the *consequent*. Adopting the symbolism explained in the Scholium to II. 1, the two terms of a ratio may be represented by a, b, or any other two letters of the alphabet, and their ratio may be expressed by writing $a : b$ (which is read) a is to b; or by $\frac{a}{b}$ which is read a divided by b; thus, if a represented 15 and b 5, then $\frac{a}{b}$ is the same as $\frac{15}{5}$, or as 15 divided by 5, namely 3, which is the *measure* of the ratio of the two quantities represented by a and b.

4. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

5. The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third are taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of

the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

SCHOLIUM. To render this definition as clear as possible, it may be symbolically expressed as follows:—Let A, B, C, and D represent four magnitudes, then, the first A is said to have the same ratio to the second B, which the third C has to the fourth D, when if A and C are multiplied by any number whatever as m, and B and D are multiplied by any other number, as n, it is found, that

If mA be $< nB$, then mC is $< nD$,
if $mA = nB$, then $mC = nD$,
or if mA be $> nB$, then mC is $> nD$.

6. Magnitudes which have the same ratio are called proportionals.

SCHOLIUM. The *arithmetical* definition of proportion is as follows:—Four quantities are said to be proportional, or in proportion, when the quotient of the first divided by the second is equal to the quotient of the third divided by the fourth, whether these quotients be either integers or fractions.

Thus $\frac{15}{5} = 3$, and $\frac{9}{3} = 3$, therefore the numbers 15, 5, 9, and 3 are said to be *in proportion;* and this is usually expressed by writing them thus, 15 : 5 : : 9 : 3, which is read as 15 is to 5 so is 9 to 3.

Euclid's definition of proportion has been found fault with because it bears no resemblance to the common notions of the similitude of ratios employed in Arithmetic or Algebra; and with the view of removing this objection, Elrington has substituted the following, namely, "Magnitudes are said to be in the same ratio, the first to the second as the third to the fourth, when any submultiple whatsoever of the first is contained in the second, as often as an equi-submultiple of the third is contained in the fourth." On the other hand, many of the most able geometers have maintained that the fifth book of Euclid is a masterpiece of skilful reasoning; and that none of the attempts which have been made to supersede it, have been successful in preserving the same unbroken chain of strict geometrical reasoning.

This objection, however, to Euclid's method of treating proportion, may be, to a great extent, removed by comparing his definition with the arithmetical one just given, and by showing that both lead to the same results. We have already explained that all species of geometrical magnitude may be expressed by letters and numbers, and we shall therefore proceed to illustrate and explain Euclid's definition by reasoning drawn from the properties of proportional numbers. We have just stated that four numbers are considered proportionals when the quotient arising from the division of the first by the second is equal to that arising from the division of the third by the fourth. Now in performing this division it may happen that the second term is not exactly contained in the first, but that a certain remainder is left; in such case we multiply this remainder by 10, and again divide by the second term, and if a fresh remainder arises, we again multiply it by 10 and repeat the division, and thus proceed either until no remainder is left, or until the remainder is too small to be of any consequence. And if instead of numbers we had two magnitudes (A and B) to deal with, we should proceed in a manner precisely similar, for, supposing B to be the lesser, we should, by continual subtraction of B from A until a magnitude was left less than B, determine how often B was contained in A; the remaining magnitude we should then increase, say 10 times, and again subtract B until another remainder less than B was obtained, which should be again increased by 10, and the process continued until a sufficiently accurate result had been obtained. The series of products thus obtained should then be ranged in order, placing first the number of times that B was contained in A, then in the first remainder, then in the second, and so on through the whole series. And it is obvious that the process which we have described may be performed with any two magnitudes of the same kind, whether lines, surfaces, solids, or angles.

Now if in place of two magnitudes we have four, A, B, C, and D, and upon dividing A by B, and C by D, we in both cases obtain identical results, that is to say, that the two serieses of products, derived from the division of A by B, and of C by D, when arranged in similar order shall be identical, then the four magnitudes which A, B, C, and D represent will be in proportion.

Now if in place of multiplying any successive number of remainders by 10, the magnitude to be divided had, in the first instance, been multiplied by the product of that number of tens, and then divided by the second magnitude, the quotient obtained would be identical with that already derived by the first process. Thus, if instead of three successive remainders having been multiplied by 10, and the division subsequently performed upon them, the first magnitude had been multiplied by the product of 3 tens, or by 1000, and then the division performed, no difference would be found in the quotient obtained. Therefore our test for the proportionality of the four magnitudes may be thus expressed:—If the first, when multiplied any number of times by 10, and then divided by the second, gives the same quotient as the third multiplied the same number of times by 10, and divided by the fourth, the four magnitudes are proportional.

Again, it must be evident that any number might be substituted for 10, which has only been adopted in the foregoing explanation, because its use is familiar in arithmetic. And our test may therefore be generalized as follows:—If the first multiplied by *any* number, and divided by the second, gives the same quotient as the third multiplied by the same number, and divided by the fourth, the four magnitudes are proportional Or, to bring it still nearer to the language of Euclid's definition:—The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, the second is contained as often in the equimultiple of the first, as the fourth is contained in the equimultiple of the third.

Now let A, B, C, D, be four magnitudes determined to be in proportion by the test just given; let m be the number by which the first and third are to be multiplied, and n the quotient derived by the subsequent division by the other terms. Since, therefore, A, B, C, and D are proportional,

$$\frac{m\mathrm{A}}{\mathrm{B}} = \frac{m\mathrm{C}}{\mathrm{D}} = n.$$

But if $\frac{m\,\mathrm{A}}{\mathrm{B}} = n$, $m\,\mathrm{A} = n\,\mathrm{B}$, and, similarly, $m\,\mathrm{C} = n\,\mathrm{D}$.

If, however, we suppose that $\frac{m\,\mathrm{A}}{\mathrm{B}}$ is not exactly equal to n, but is somewhat less, then as $\frac{m\,\mathrm{A}}{\mathrm{B}}$ and $\frac{m\,\mathrm{C}}{\mathrm{D}}$ are equal, $\frac{m\,\mathrm{C}}{\mathrm{D}}$ is also somewhat less than n, and therefore $m\,\mathrm{A}$ is $<$ $n\,\mathrm{B}$, and $m\,\mathrm{C}$ is $<$ $n\,\mathrm{D}$.

Again, let $\frac{m\,A}{B}$ be somewhat more than n, then also so is $\frac{m\,C}{D}$ somewhat more than n, and therefore m A is $>$ n B, and m C is $>$ n D.

Now, collecting these results we have

$$\begin{aligned}&\text{If } m\,\text{A be} < n\,\text{B, then } m\,\text{C is} < n\,\text{D},\\ &\text{if } m\,\text{A} = n\,\text{B, then } m\,\text{C} = n\,\text{D},\\ &\text{or if } m\,\text{A is} > n\,\text{B, then } m\,\text{C is} > n\,\text{D}.\end{aligned}$$

which being compared with Euclid's definition, as symbolically expressed at page 18, will be found to be identical.

That the definition of proportion here given by Euclid was only meant to be applied to geometrical quantities, is evident from the fact that he has given another for proportional numbers in the seventh book; but it should be observed that all his conclusions may be generalized so as to apply with equal truth, in the case of numbers, by the substitution of the word "quantity" for "magnitude."

The perfection of Euclid's method is, that one demonstration suffices either when $\frac{m\,A}{B} = n$, is $>$ than n, or is $<$ than n, whereas, with all other methods, when rigorous proof is requisite, they require two demonstrations to each proposition, one when $\frac{m\,A}{B} = n$, and another when $\frac{m\,A}{B}$ is $>$ or $<$ n; and this latter case has usually to be proved from the former by a "reductio ad absurdum."

It should be observed that, in any proportion, the first and second terms must be of the same kind, and the third and fourth of the same kind, but the two pairs may differ; thus, the first and second magnitudes may be two lines or angles, while the third and fourth are surfaces or solids.

7. When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

8. Analogy or proportion is the equality of ratios.

In this definition the term "equality" has been substituted for "similitude," the word employed by Euclid. The whole definition might have been omitted, as being unnecessary.

9. Proportion consists in three terms at least.

This is rather an inference than a definition. Three quantities may form a proportion when the middle term is both the consequent of the first ratio and the antecedent of the second; thus, when A : B : : B : C. In such a case B is termed a *mean proportional* between A and C. When a series of quantities are such that each middle term is the consequent of that which precedes it, and the antecedent of that which follows it, or when, in other words, every term bears an equal ratio to that which follows it, such a series is said to be in *continued* proportion. In any proportion the first and last terms are called the *extremes*, and all the others the *mean terms*.

10. When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

11. When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

12. When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D, be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C, the same ratio that G has to H; and C to D, the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L: and the same thing is to be understood when it is more briefly expressed, by saying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

Arithmetically ratios are compounded by multiplying together all the antecedents of the separate ratio for a new antecedent, and all the consequents together for a new consequent. Thus the ratio 120 : 960 is compounded of the ratios 3 : 6, 5 : 10, and 8 : 16, for $3 \times 5 \times 8 = 120$, and $6 \times 10 \times 16 = 960$.

A *duplicate ratio* is that which is compounded of *two* equal ratios, as of A : B, B : C; a *triplicate ratio* is compounded of *three* equal ratios, as of A : B, B : C, C : D; a *quadruplicate ratio*, is compounded of *four* equal ratios; a *quintuplicate* of *five* equal ratios, and so on.

Thus, if A, B, C, be in continued proportion, then

$$A : B :: B : C$$

$$\text{and } \frac{A}{B} = \frac{B}{C}, \text{ also } \frac{A}{B} = \frac{A}{B};$$

then multiplying these two equations together

$$\frac{A}{B} \times \frac{A}{B} = \frac{B}{C} \times \frac{A}{B};$$

$$\text{or, } \frac{A}{C} = \frac{A^2}{B^2};$$

that is, A : C : : $A^2 : B^2$, or A is to C in the *duplicate* ratio of A to B.

Again, if A, B, C, D, be in continued proportion,

$$\frac{A}{B}, \frac{B}{C}, \frac{C}{D} \text{ are all equal;}$$

and $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B}$;

therefore $\frac{A}{D} = \frac{A^3}{B^3}$;

or A is to D in the *triplicate* ratio of A to B, and so on with any number of quantities in continued proportion.

13. **In proportionals, the antecedent terms are called homologous to one another, as also the antecedents to one another.**

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of proportionals, so as that they continue still to be proportionals.

14. *Permutando*, or *alternândo*, by permutation, or alternately; this word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third, which the second has to the fourth; or that the first is to the third, as the second to the fourth: as is shown in the 16th proposition of this book.

15. *Invertendo*, by inversion; when there are four proportionals, and it is inferred, that the second is to the first, as the fourth to the third. Proposition B, book 5.

16. *Componendo*, by composition; when there are four proportionals, and it is inferred, that the first, together with the second, is to the second, as the third together with the fourth, is to the fourth. Proposition XVIII., book 5.

17. *Dividendo*, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. Proposition XVII., book 5.

18. *Convertendo*, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Proposition E, book 5.

19. *Ex æquali* (sc. *distantiâ*), or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: "Of this there are the two following kinds, which

arise from the different order in which the magnitudes are taken two and two."

20. *Ex æquali*, from equality; this term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in Proposition XXII., book 5.

21. *Ex æquali, in proportione; perturbatâ, seu inordinatâ*, from equality, in perturbate or disorderly proportion (*Prop.* 4, *Lib. II. Archimedis de sphæra et cylindro*); this term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank: and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank: and so on in a cross order: and the inference is as in the 19th definition. It is demonstrated in Proposition XXIII., book 5.

The following table will serve to illustrate and explain the foregoing seven last definitions.

If any four magnitudes be in proportion,

so that $A : B :: C : D$

Then *Permutando* or *Alternando* $A : C :: B : D$

Invertendo $B : A :: D : C$

Componendo $A + B : B :: C + D : D$

Dividendo $A - B : B :: C - D : D$

Convertendo $\left\{\begin{array}{l} A : A + B :: C : C + D \\ A : A - B :: C : C - D \end{array}\right.$

Further, if there be four magnitudes, so that

$A : B :: C : D$

and four others, so that $L : M :: N : P$

Then *ex æquali*,

if $A : B :: L : M$

$B : C :: M : N$

$C : D :: N : P$

then $A : D :: L : P$.

And *ex æquali, in proportione perturbatâ*,

if $A : B :: N : P$

$B : C :: M : N$

$C : D :: L : M$

then $A : D :: L : P$

The terms *subduplicate*, *subtriplicate*, and *sesquiplicate ratios* being frequently employed in astronomy should be defined.

If *three* quantities be in continued proportion, the first is said to have to the second the *subduplicate ratio* of that which the first has to the third.

Thus, if A, B, C, are in continued proportion, A is said to have to B the *subduplicate ratio* of that which A has to C, and may be expressed algebraically

$$A : B :: A^{\frac{1}{2}} : C^{\frac{1}{2}}.$$

If *four* quantities be in continued proportion, the first is said to have to the second the *subtriplicate ratio* of that which the first has to the fourth.

Thus, if A, B, C, D, are in continued proportion, A is said to have to B the *subtriplicate ratio* of that which A has to D, and may be expressed algebraically, $A : B :: A^{\frac{1}{3}} : D^{\frac{1}{3}}$.

A *sesquiplicate ratio* is the ratio compounded of the simple ratio and the subduplicate, and may be expressed algebraically, $A : B :: A^{\frac{3}{2}} : C^{\frac{3}{2}}$.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

Or if equals be *multiplied* by the same, the products are equal.

2. Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

Or if equals be *divided* by the same, the quotients are equal.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. Of two magnitudes that one of which a multiple is greater than the same multiple of the other, is the greater.

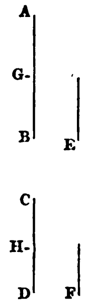
In the following propositions lines are employed by Euclid to represent proportional magnitudes, but it should be understood that any *similar* magnitudes might have been employed, such as plane figures, solid bodies, or angles.

PROPOSITION I.

Theorem.—*If any number of magnitudes be equimultiples of as many others, each of each;* what multiple soever any one of the first is of its part, the same multiple shall all the first magnitudes taken together be of all the others taken together.

Let any number of magnitudes AB, CD, be equimultiples of as many others E, F, each of each; whatsoever multiple AB is of E, the same multiple shall AB and CD together be of E and F together.

Demonstration. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD, equal each of them to F; the number, therefore, of the magnitudes CH, HD is equal to the number of the others AG, GB (*a*). And because AG is equal to E, and CH to F, therefore AG and CH together are equal to E and F together (*b*). For the same reason, GB and HD together are equal to E and F together; wherefore, as many magnitudes as are in AB equal to E, so many are there in AB and CD together equal to E and F together. Therefore, whatever multiple AB is of E, the same multiple are AB and CD together, of E and F together; and the same demonstration would hold if the number of magnitudes were greater than two. Therefore, if any number of magnitudes, &c.



(*a*) Hypoth.
(*b*) I. Ax. 2.

Scholium. In order to the elucidation of Euclid's demonstrations we shall append to each proposition an algebraical investigation and proof, preserving his train of reasoning unaltered.

Theorem. *If* A, B, C, *&c., be equimultiples of* a, b, c, *&c.*, then whatsoever multiple A is of a, the same multiple is $A + B + C +$ &c., of $a + b + c +$ &c.

Let A contain n parts each equal to a, then

$$A = n a$$

and because B, C, &c., are the same multiples of b, c, &c., that A is of a, therefore

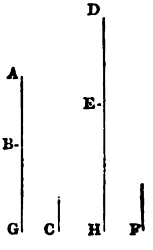
$$B = n b$$
$$C = n c$$
&c.

$$\begin{aligned}
\text{therefore } A + B + C + \&c. &= n\,a + n\,b + n\,c + \&c.\\
&= a + a + a + \&c. \text{ to } n \text{ terms}\\
&\quad + b + b + b + \&c. \text{ to } n \text{ terms}\\
&\quad + c + c + c + \&c. \text{ to } n \text{ terms}\\
&= (a + b + c + \&c.) + (a + b + c + \&c.) \text{ to } n \text{ terms}\\
&= n\,(a + b + c + \&c.)
\end{aligned}$$

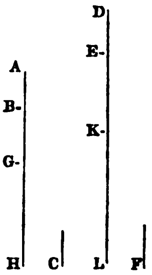
PROPOSITION II.

THEOREM.—*If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth;* then shall the first, together with the fifth, be the same multiple of the second, that the third, together with the sixth, is of the fourth.

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth, and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth; then shall AG the first together with the fifth, be the same multiple of C the second, that DH the third together with the sixth, is of F the fourth.



DEMONSTRATION. Because AB is the same multiple of C that DE is of F, there are as many magnitudes in AB equal to C as there are in DE equal to F; in like manner, as many as there are in BG equal to C, so many are there in EH equal to F; therefore, as many as there are in the whole AG equal to C, so many are there in the whole DH equal to F; therefore, AG is the same multiple of C that DH is of F, that is, AG, the first and fifth together, is the same multiple of the second C, that DH, the third and sixth together, is of the fourth E. Therefore, if the first magnitude, &c.



Corollary. From this it is evident that if any number of magnitudes AB, BG, GH, be multiples of another C, and as many DE, EK, KL, be the same multiples of F, each of each; the whole of the first, viz. AH, is the same multiple of C, that the whole of the last, viz. DL, is of F.

Scholia. 1. This proposition, algebraically expressed, is as follows:—

Theorem. *If* A, a, B, b, A_1, B_1, *be six magnitudes such that* A, B *are equimultiples of a and b, and* A_1, B_1, *are also equimultiples of a and b,* then $A + A_1$, $B + B_1$ shall be equimultiples of a and b.

Let A contain a, m times,
and A_1 contain a, n times;
then also B will contain b, m times,
and B_1 will contain b, n times.

Therefore,

$$A = m \,.\, a, \qquad B = m \,.\, b,$$
$$A_1 = n \,.\, a, \qquad B_1 = n \,.\, b;$$

and adding equals together,

$$A + A_1 = m \,.\, a + n \,.\, a = (m + n) \,.\, a,$$
$$\text{and } B + B_1 = m \,.\, b + n \,.\, b = (m + n) \,.\, b;$$

that is, $A + A_1$ and $B + B_1$ are equimultiples of a and b.

2. The corollary may be algebraically expressed as follows:—

$$\text{If } A = m.\, a,\ A_1 = n.\, a,\ A_2 = p.\, a,\ A_3 = q.\, a, \text{ \&c.}$$
$$\text{and } B = m.\, b,\ B_1 = n.\, b,\ B_2 = p.\, b,\ B_3 = q.\, b, \text{ \&c.}$$

then,

$$A + A_1 + A_2 + A_3 + \text{\&c.} = m \,.\, a + n \,.\, a + p \,.\, a + q \,.\, a + \text{\&c.} = (m + n + p + q + \text{\&c.})\ a.$$
$$\text{and } B + B_1 + B_2 + B_3 + \text{\&c.} = m \,.\, b + n \,.\, b + p \,.\, b + q \,.\, b + \text{\&c.} = (m + n + p + q + \text{\&c.})\ b.$$

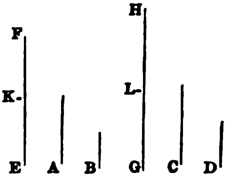
Therefore, $A + A_1 + A_2 +$ &c., $B + B_1 + B_2 +$ &c. are equimultiples of a and b.

PROPOSITION III.

Theorem.—*If the first be the same multiple of the second which the third is of the fourth, and if of the first and third there be taken equimultiples,* these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second that

C the third is of D the fourth; and of A and C let the equimultiples EF and GH be taken, then EF is the same multiple of B that GH is of D.



(a) V. 2.

Demonstration. Because EF is the same multiple of A that GH is of C, there are as many magnitudes in EF equal to A as there are in GH equal to C; let EF be divided into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C, therefore the number of the magnitudes EK, KF, shall be equal to the number of the others GL, LH; and because A is the same multiple of B that C is of D, and that EK is equal to A, and GL equal to C, therefore EK is the same multiple of B that GL is of D; for the same reason, KF is the same multiple of B that LH is of D, and the same holds if there be more parts in EF, GH, equal to A, C; therefore, because the first EK is the same multiple of the second B which the third GL is of the fourth D, and that the fifth KF is the same multiple of the second B which the sixth LH is of the fourth D; EF, the first together with the fifth, is the same multiple of the second B which GH, the third together with the sixth, is of the fourth D (*a*). Therefore, if the first be the same multiple, &c.

Scholium. The foregoing proposition, algebraically expressed, is as follows:—

Theorem. *If of four magnitudes the first* A *is the same multiple of the second* a, *which the third* B *is of the fourth* b, *and if of* A *and* B *equimultiples be taken*, these shall also be equimultiples of a and b.

Let A contain a, m times,
and B contain b, m times; then

$$A = m \,.\, a, \text{ and } B = m \,.\, b;$$

and if the equimultiples of A and B be taken such that they shall contain A and B, n times, they shall be respectively

$$n \,.\, A, \text{ and } n \,.\, B\,.$$

Now because A and B contain a and b, m times, therefore n . A and n . B contain a and b, $m\,n$ times, and

$$n \,.\, A = n \,.\, m \,.\, a$$
$$\text{and } n \,.\, B = n \,.\, m \,.\, b$$

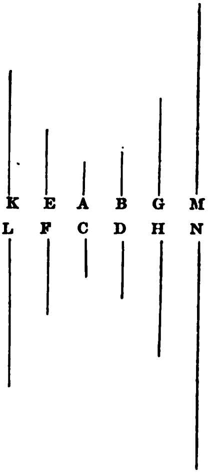
that is, n . A, n . B are equimultiples of a and b.

PROPOSITION IV.

THEOREM.—*If the first of four magnitudes has the same ratio to the second which the third has to the fourth,* then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, *i. e.* "the equimultiple of the first shall have the same ratio to that of the second which the equimultiple of the third has to that of the fourth."

Let A the first have to B the second the same ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E, F; and of B and D any equimultiples whatever G, H; then shall E have the same ratio to G that F has to H.

DEMONSTRATION. Take of E and F any equimultiples whatever K, L; and of G, H, any equimultiples whatever M, N; then because E is the same multiple of A that F is of C; and of E and F the equimultiples K, L, have been taken; therefore K is the same multiple of A that L is of C (*a*); for the same reason, M is the same multiple of B that N is of D. And because, as A is to B, so is C to D (*b*), and of A and C have been taken certain equimultiples K, L, and of B and D have been taken certain equimultiples M, N; therefore if K be greater than M, L is greater than N; and if equal, equal; if less, less (*c*); but K, L are any equimultiples whatever of E, F, and M, N any whatever of G, H; therefore as E is to G so is F to H (*c*).



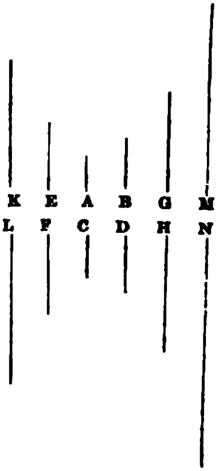
COROLLARY. Likewise, *if the first has the same ratio to the second which the third has to the fourth,* then also any equimultiples whatever of the first and third shall have the same ratio to the

(*a*) V. 3.
(*b*) Hypoth.
(*c*) V. Def. 5.

second and fourth; and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second the same ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E shall be to B as F to D.

DEMONSTRATION. Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; then it may be demonstrated, as before, that K is the same multiple of A that L is of C; and because A is to B as C is to D (*b*), and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples G, H; therefore if K be greater than G, L is greater than H; and if equal, equal; if less, less (*c*); but K, L are any equimultiples whatever of E, F, and G, H any whatever of B, D; therefore, as E is to B, so is F to D (*c*). And in a similar way the other case is demonstrated.



(*b*) Hypoth.
(*c*) V. Def. 5.

SCHOLIUM. The fourth proposition may be algebraically expressed as follows:—

THEOREM. *If* $A : a :: B : b$; then $m A : m B :: n a : n b$.

Because $A : a :: B : b$

$$\frac{A}{a} = \frac{B}{b};$$

multiplying both sides by a, and dividing both sides by B,

$$\frac{A}{B} = \frac{a}{b}$$

$$\text{and } \frac{m \,.\, A}{m \,.\, B} = \frac{n \,.\, a}{n \,.\, b},$$

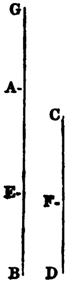
therefore $m \,.\, A : m \,.\, B :: n \,.\, a : n \,.\, b$.

If n be taken equal to unity, the above will be a demonstration of the corollary.

PROPOSITION V.

THEOREM.—*If one magnitude be the same multiple of another which a part taken from the first is of a part taken from the other*, the first remainder is the same multiple of the second that the first magnitude is of the second.

Let the magnitude AB be the same multiple of CD that AE taken from the first is of CF taken from the other; the remainder EB shall be the same multiple of the remainder FD that the whole AB is of the whole CD.



DEMONSTRATION. Take AG the same multiple of FD that AE is of CF; therefore, AE is the same multiple of CF that EG is of CD (*a*); but AE is the same multiple of CF that AB is of CD (*b*); therefore EG is the same multiple of CD that AB is of CD; wherefore EG is equal to AB (*c*); take from each of them the common magnitude AE; and the remainder AG is equal to the remainder EB. Wherefore, since AE is the same multiple of CF that AG is of FD, and that AG is equal to EB; therefore AE is the same multiple of CF that EB is of FD; but AE is the same multiple of CF that AB is of CD (*b*); therefore EB is the same multiple of FD that AB is of CD. Therefore, if one magnitude, &c.

(*a*) V. 1.
(*b*) Hypoth.
(*c*) V. Ax. 1.

SCHOLIUM. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If* A *is the same multiple of* a *that* B *is of* b, then A − B is the same multiple of $a - b$.

For let

$$A = m \,.\, a, \text{ and } B = m \,.\, b,$$

then,

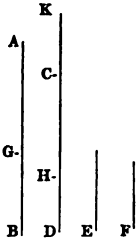
$$A - B = m \,.\, a - m \,.\, b = m \,.\, (a - b).$$

PROPOSITION VI.

THEOREM.—*If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two*, the remainders are either equal to these others, or equimultiples of them.

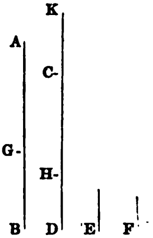
Let the two magnitudes AB, CD, be equimultiples of the two E, F, and let AG, CH, taken from the first two be equimultiples of the same E, F; the remainder GB, HD, shall be either equal to E, F, or equimultiples of them.

Demonstration. First, let GB be equal to E: HD shall be equal to F. Make CK equal to F: and because AG is the same multiple of E that CH is of F (*a*), and that GB is equal to E, and CK to F; therefore AB is the same multiple of E that KH is of F; but AB is the same multiple of E that CD is of F; therefore KH is the same multiple of F that CD is of F; wherefore KH is equal to CD (*b*); take away the common magnitude CH, then the remainder CK is equal to the remainder HD; but CK is equal to F; therefore HD is equal to F.



(*a*) Hypoth.
(*b*) V. Ax. 1.

Next let GB be a multiple of E; HD shall be the same multiple of F. Make CK the same multiple of F that GB is of E; and because AG is the same multiple of E that CH is of F (*a*); and GB the same multiple of E that CK is of F; therefore AB is the same multiple of E that KH is of F (*c*); but AB is the same multiple of E that CD is of F (*a*); therefore KH is the same multiple of F that CD is of F; wherefore KH is equal to CD (*b*); take away CH from both, and the remainder KC is equal to the remainder HD; and because GB is the same multiple of E that KC is of F, and that KC is equal to HD; therefore HD is the same multiple of F that GB is of E.



(*c*) V. 2.

Scholium. The foregoing proposition, algebraically expressed, is as follows:—

Theorem. *If* A, B *be equimultiples of a and b*, then $A - m \,.\, a$, $B - m \,.\, b$ are either equimultiples of a, b, or are equal to them.

$$\text{Let } A = n \,.\, a, \text{ and } B = n \,.\, b;$$

$$\text{then } A - m \,.\, a = n \,.\, a - m \,.\, a = (n - m)\, a,$$
$$\text{and } B - m \,.\, b = n \,.\, b - m \,.\, b = (n - m)\, b.$$

which is the second case in Euclid; when $n = 2$, and $m = 1$

$$A - m \,.\, a = a, \text{ and } B - m \,.\, b = b,$$

which is the first case in Euclid.

PROPOSITION A.

THEOREM.—*If the first of four magnitudes have the same ratio to the second which the third has to the fourth*, then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

DEMONSTRATION. Take any equimultiples of each of them, as the doubles of each; then by Def. 5 of this book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth; but if the first be greater than the second, the double of the first is greater than the double of the second; wherefore, also, the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth; in like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth, or less than it.

SCHOLIUM. This proposition and the three following have been added by Simson. It may be expressed algebraically as follows:—

THEOREM. *If* $A : a :: B : b$, then, according as A is $>$, $=$, or $< a$ B is $>$, $=$, or $< b$.

Let any equimultiples of them be taken, as

$$m \,.\, A,\ m \,.\, a,\ m \,.\, B,\ m \,.\, b\,;$$

then by V. Def. 5, according as

$$m \,.\, A \text{ is } >,\ =, \text{ or } <\ m \,.\, a,\ m \,.\, B \text{ is } >,\ =, \text{ or } <\ m \,.\, b.$$

$$\text{But if } A \text{ be } >,\ =, \text{ or } <\ a, \text{ then } m \,.\, A \text{ is } >,\ =, \text{ or } <\ m \,.\, a\,;$$

$$\text{therefore } m \,.\, B \text{ is } >,\ =, \text{ or } <\ m \,.\, b,$$

$$\text{and } B \text{ is } >,\ =, \text{ or } <\ b.$$

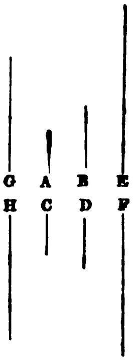
$$\text{Therefore, according as } A \text{ is } >,\ =, \text{ or } <\ a,\ B \text{ is } >,\ =, \text{ or } <\ b.$$

PROPOSITION B.

THEOREM.—*If four magnitudes are proportionals,* they are proportionals also when taken inversely.

If the magnitude A be to B as C is to D, then also inversely B is to A as D to C.

Demonstration. Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. First, let E be greater than G, then G is less than E; and because A is to B as C is to D (*a*), and of A and C, the first and third, G and H are equimultiples; and of B and D, the second and fourth, E and F are equimultiples; and that G is less than E, therefore H is less than F (*b*); that is, F is greater than H; if, therefore, E be greater than G, F is greater than H; in like manner, if E be equal to G, F may be shown to be equal to H; and if less, less; but E, F, are any equimultiples whatever of B and D, and G, H, any whatever of A and C; therefore, as B is to A so is D to C (*b*).



(*a*) Hypoth.
(*b*) V. Def. 5.

Scholium. This proposition, algebraically expressed, is as follows:—

Theorem.—*If* A : *a* :: B : *b*, then *a* : A :: *b* : B.

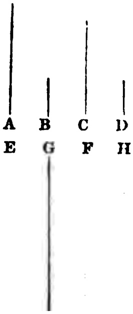
$$\text{For } \frac{A}{a} = \frac{B}{b}:$$

$$\text{therefore } \frac{a}{A} = \frac{b}{B};$$

and therefore *a* : A :: *b* : B.

PROPOSITION C.

Theorem.—*If the first be the same multiple or submultiple of the second that the third is of the fourth,* the first is to the second as the third is to the fourth.

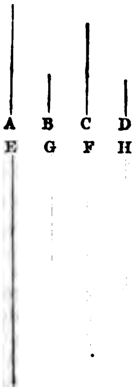


Let the first A be the same multiple of the second B that the third C is of the fourth D; A is to B as C is to D.

Demonstration. Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H; then, because A is the same multiple of B that C is of D (*a*); and that E is the same multiple of A that F is of C; therefore E is the same mul-

(*a*) Hypoth.

tiple of B that F is of D (*b*); that is, E and F are equimultiples of B and D; but G and H are equimultiples of B and D; therefore, if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H; in like manner, if E be equal to G, or less than it, F may be shown to be equal to H, or less than it; but E, F are any equimultiples whatever of A, C; and G, H any equimultiples whatever of B, D; therefore A is to B as C is to D (*c*).



Next, let the first A be the same submultiple of the second B that the third C is of the fourth D; A shall be to B as C is to D.

For since A is the same submultiple of B that C is of D, therefore B is the same multiple of A that D is of C; wherefore, by the preceding case, B is to A as D is to C; and therefore inversely, A is to B as C is to D (*d*).

(*b*) V. 3.
(*c*) V. Def. 5.

(*d*) V. B.

Scholium. The foregoing proposition, expressed algebraically, is as follows:—

Theorem.—*If* $A = m \,.\, a$, *and* $B = m \,.\, b$, *or if* $A = \frac{a}{m}$, *and* $B = \frac{b}{m}$, then $A : a :: B : b$.

For, in the first case,

$$\frac{A}{a} = m, \text{ and } \frac{B}{b} = m,$$

therefore,

$$\frac{A}{a} = \frac{B}{b}.$$

and therefore $A : a :: B : b$.

In the second case,

$$\frac{A}{a} = \frac{1}{m}, \text{ and } \frac{B}{b} = \frac{1}{m},$$

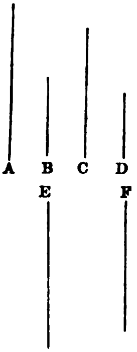
$$\text{therefore } \frac{A}{a} = \frac{B}{b},$$

and $A : a :: B : b$.

PROPOSITION D.

Theorem.—*If the first be to the second as the third to the fourth, and if the first be a multiple or submultiple of the second,* the third is the same multiple or submultiple of the fourth.

Let A be to B as C is to D; and first let A be a multiple of B, then C shall be the same multiple of D.

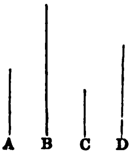


Demonstration. Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D; then, because A is to B as C is to D (*a*); and of B the second, and D the fourth, equimultiples have been taken, E and F; therefore A is to E as C is to F (*b*); but A is equal to E, therefore C is equal to F (*c*); and F is the same multiple of D that A is of B: therefore C is the same multiple of D that A is of B.

Next, let A be a submultiple of B; then C shall be the same submultiple of D.

Because A is to B as C is to D (*a*); then inversely, B is to A as D is to C (*d*); but A is a submultiple of B, that is, B is a multiple of A; therefore, by the preceding case, D is the same multiple of C; that is, C is the same submultiple of D that A is of B.

(*a*) Hypoth.
(*b*) V. cor. 4.
(*c*) V. A.



(*d*) V. B.

Scholium. This proposition is the inverse of the preceding; it may be algebraically expressed as follows :—

Theorem. *If* $A : a :: B : b$, *and* $A =$ *either* $m \, . \, a$ *or* $\frac{a}{m}$, then $B = m \, . \, b$, or $\frac{b}{m}$.

For

$$\frac{B}{b} = \frac{A}{a},$$

and because

$$\frac{A}{a} = m, \text{ or } \frac{1}{m},$$

$$\text{therefore } \frac{B}{b} = m, \text{ or } \frac{1}{m},$$

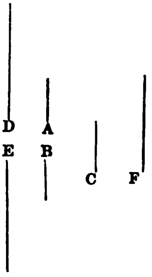
and multiplying by b,

$$B = m \, . \, b, \text{ or } \frac{b}{m}.$$

PROPOSITION VII.

THEOREM.—*Equal* magnitudes have the same ratio to the same magnitude; and the same has the same ratio to *equal* magnitudes.

Let A and B be equal magnitudes, and C any other. A and B shall each of them have the same ratio to C: and C shall have the same ratio to each of the magnitudes A and B.



DEMONSTRATION. Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F: then, because D is the same multiple of A that E is of B, and that A is equal to B (*a*): therefore D is equal to E (*b*): therefore if D be greater than F, E is greater than F; and if equal, equal; if less, less; but D, E, are any equimultiples of A, B, and F is any multiple of C; *therefore* A *is to* C *as* B *is to* C (*c*).

Likewise C has the same ratio to A, that it has to B, or C is to A as C is to B. For, having made the same construction, D may in like manner be shown to be equal to E; therefore if F be greater than D, it is likewise greater than E; and if equal, equal; if less, less; but F is any multiple whatever of C, and D, E, are any equimultiples whatever of A, B; *therefore* C *is to* A *as* C *is to* B (*c*).

(*a*) Hypoth.
(*b*) V. Ax. 1.
(*c*) V. Def. 5.

SCHOLIUM. This proposition, algebraically expressed, is as follows:—

THEOREM. *If* A = B, *and* C *be any third quantity*, A : C :: B : C, and C : A :: C : B.

Since A = B,

$$\frac{A}{C}=\frac{B}{C},$$

therefore A : C :: B : C.

$$\text{Also } \frac{C}{A}=\frac{C}{B},$$

therefore C : A :: C : B

PROPOSITION VIII.

THEOREM.—*If two magnitudes are unequal,* **the greater has a greater ratio to any other magnitude than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.**

Let AB, BC be two unequal magnitudes, of which AB is the greater, and let D be any other magnitude. AB shall have a greater ratio to D than BC has to D: and D shall have a greater ratio to BC than it has to AB.

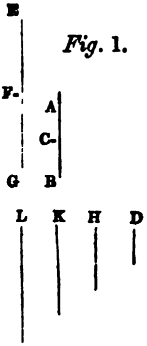


Fig. 1.

DEMONSTRATION. If the magnitude which is not the greater of the two AC, CB be not less than D, take EF, FG, the doubles of AC, CB (as in Fig. 1). But if that which is not the greater of the two AC, CB be less than D (as in Figs. 2 and 3), this magnitude can be multiplied, so as to become greater than D, whether it be AC or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the same multiple of CB: therefore EF and FG are each of them greater than D: and in every one of the cases, take H the double of D, K its triple, and so on, till the multiple of D be that which first becomes greater than FG: let L be that multiple of D which is first greater than FG, and K the multiple of D which is next less than L.

(a) V. 1.
(b) V. Def. 7.

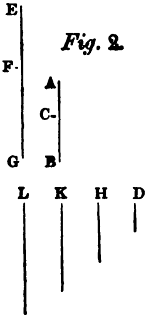


Fig. 2.

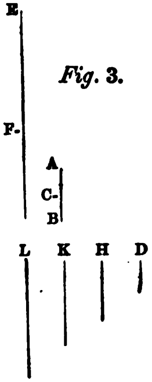


Fig. 3.

Then, because L is the multiple of D which is the first that becomes greater than FG, the next preceding multiple K is not greater than FG; that is, FG is not less than K: and since EF is the same multiple of AC that FG is of CB; therefore FG is the same multiple of CB that EG is of AB (*a*): that is, EG and FG are equimultiples of AB and CB: and since it was shown that FG is not less than K, and by the construction EF is greater than D; therefore the whole EG is greater than K and D together: but K together with D is equal to L; therefore EG is greater than L: but FG is not greater than L: and EG, FG were proved to be equimultiples of AB, BC; and L is a multiple of D; *therefore* AB *has to* D *a greater ratio than* BC *has to* D (*b*).

Also D shall have to BC a greater ratio than it has to AB. For having made the same construction, it may be shown, in like manner, that L is greater than FG, but that it is not greater than EG; and L is a multiple of D; and FG, and EG were proved to be equimultiples of CB, AB; *therefore* D *has to* CB *a greater ratio than it has to* AB (*b*).

SCHOLIUM. This proposition may be algebraically expressed as follows:—

THEOREM. *If* A *is* $>$ B, then A : C is $>$ B : C, and C : B is $>$ C : A.

For if A is $>$ B,

$$\frac{A}{C} \text{ is} > \frac{B}{C};$$

and therefore A : C is $>$ B : C.

Also if A is $>$ B,

$$\frac{C}{B} \text{ is} > \frac{C}{A},$$

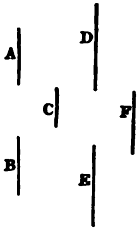
and therefore C : B is $>$ C : A.

PROPOSITION IX.

THEOREM.—*If magnitudes have the same ratio to the same magnitude,* they are equal to one another: *and those to which the same magnitude has the same ratio* are equal to one another.

Let A, B have each of them the same ratio to C; then A shall be equal to B.

DEMONSTRATION. For, if they are not equal, one of them must be greater than the other: let A be the greater: then, by what was shown in the preceding proposition, there are some equimultiples of A and B, and some multiple of C such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let these multiples be taken; and let D, E be the multiples of A, B, and F the multiple of C, such that D may be greater than F, but E not greater than F: then, because A is to C as B is to C, and of A, B are taken equimultiples, D, E, and of C is taken a multiple F; and that D is greater than F; therefore E is also greater than F (*a*): but E is not greater than F; which is impossible: *therefore* A *and* B *are not unequal; that is, they are equal.*



(*a*) V. Def. 5.

Next, let C have the same ratio to each of the magnitudes A and B; then A shall be equal to B.

For, if they are not equal, one of them must be greater than the other: let A be the greater: therefore, as was shown in the eighth proposition, there is some multiple F of C, and some equimultiples E and D of B and A, such that F is greater than E, but not greater than D: and because C is to B as C is to A, and that F the multiple of the first is greater than E the multiple of the second (*a*); therefore F the multiple of the third is greater than D the multiple of the fourth: but F is not greater than D; which is impossible. *Therefore* A *is equal to* B.

SCHOLIUM. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If* $A : B :: C : B$, then $A = C$; and *if* $B : A :: B : C$, then also $A = C$.

For if $A : B :: C : B$

$$\frac{A}{B} = \frac{C}{B};$$

therefore multiplying by B,

$$A = C.$$

Again, if $B : A :: B : C$

$$\frac{B}{A} = \frac{B}{C};$$

therefore dividing by B,

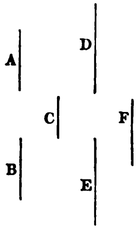
$$\frac{1}{A} = \frac{1}{C},$$

and therefore $A = C$.

PROPOSITION X.

THEOREM.—*That magnitude which has a greater ratio than another has to the same magnitude,* is the greater of the two; *and that magnitude to which the same has a greater ratio than it has to another magnitude,* is the lesser of the two.

Let A have to C a greater ratio than B has to C; then A shall be greater than B.



DEMONSTRATION. For, because A has to C a greater ratio than B has to C, there are some equimultiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it (*a*); let them be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, such that D is greater than F; but E is not greater than F, therefore D is greater than E: and because D and E are equimultiples of A and B, and D is greater than E, *therefore* A *is greater than* B (*b*).

(*a*) V. Def. 7.
(*b*) V. Def. 4.

Next, let C have a greater ratio to B than it has to A; then B shall be less than A.

For there is some multiple F of C, and some equimultiples E and D of B and A, such that F is greater than E, but not greater than D (*a*): therefore E is less than D: and because E and D are equimultiples of B and A, and that E is less than D, *therefore* B *is less than* A (*b*).

SCHOLIUM. This proposition may be algebraically expressed as follows:—

THEOREM. *If* A : B *is* $>$ C : B, then A is $>$ C; *and if* B : A *is* $>$ B : C, then A is $>$ C.

For if A : B is $>$ C : B,

$$\frac{A}{B} \text{ is} > \frac{C}{B};$$

and multiplying by B,

$$A > C.$$

Again, if B : A is B : C,

$$\frac{B}{A} \text{ is} > \frac{B}{C},$$

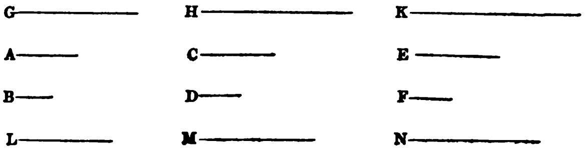
therefore CB is $>$ AB; and dividing by B,

$$C \text{ is} > A, \text{ or } A \text{ is} > C.$$

PROPOSITION XI.

THEOREM.—*If ratios are equal to the same ratio,* they are equal to one another.

Let A be to B as C is to D; and also let C be to D as E is to F; then shall A be to B as E is to F.



(*a*) V. Def. 5.

DEMONSTRATION. Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N. Therefore, since A is to B as C is to D, and G, H are taken equimultiples of A, C, and L, M, of B, D; if G be greater than L, H is greater than M; and if equal, equal; if less, less (*a*). Again, because E is to F as C is to D, and H, K are taken equimultiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; and if equal, equal; if less, less: but if G be greater than L, it has been shown that H is greater than M; and if equal, equal; if less, less: therefore if G be greater than L, K is greater than N; and if equal, equal; if less, less: and G, K are any equimultiples whatever of A, E; and L, N any whatever of B, F: *therefore A is to B as E is to F* (*a*).

SCHOLIUM. This proposition may be algebraically expressed as follows:—

THEOREM. *If* A : B :: C : D, *and* C : D :: E : F, then A : B :: E : F.
For because A : B :: C : D,

$$\frac{A}{B} = \frac{C}{D};$$

and because C : D :: E : F,

$$\frac{C}{D} = \frac{E}{F};$$

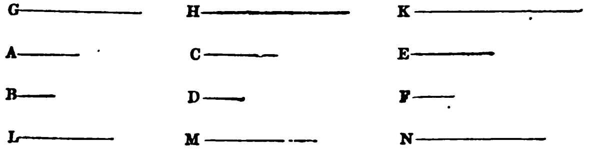
$$\text{therefore } \frac{A}{B} = \frac{E}{F},$$

and therefore A : B :: E : F

PROPOSITION XII.

Theorem.—*If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents taken together.*

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, so is C to D, and E to F: then as A is to B, so shall A, C, E together be to B, D, F together.



(*a*) V. Def. 5. (*b*) V. I.

Demonstration. Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N: then, because A is to B, as C is to D, and as E is to F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; if less, less (*a*); wherefore if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; if less, less: but G, and G, H, K together are any equimultiples of A, and A, C, E together; because if there be any number of magnitudes equimultiples of as many others, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole (*b*): for the same reason L, and L, M, N together are any equimultiples of B, and B, D, F together: *therefore as* A *is to* B, *so is* A, C, E *together to* B, D, F *together.*

Scholium. The foregoing proposition may be algebraically expressed as follows:—

Theorem. *If* $A : B :: C : D :: E : F$, &c., then $A : B :: A + C + E + \&c. : B + D + F + \&c.$

For if $A : B :: C : D :: E : F$,

$$\frac{A}{B} = \frac{C}{D} = \frac{E}{F};$$

$$\text{and therefore } \frac{A}{C} = \frac{B}{D}.$$

Adding 1 to each side,

$$\frac{A}{C} + 1 = \frac{B}{D} + 1,$$

$$\text{therefore } \frac{A + C}{C} = \frac{B + D}{D},$$

$$\text{and therefore } \frac{A + C}{B + D} = \frac{C}{D} = \frac{E}{F}.$$

Again,

$$\frac{A + C}{E} = \frac{B + D}{F},$$

adding 1 to each side,

$$\frac{A + C}{E} + 1 = \frac{B + D}{F} + 1,$$

$$\text{and } \frac{A + C + E}{E} = \frac{B + D + F}{F},$$

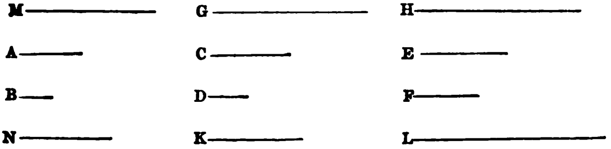
$$\text{therefore } \frac{A + C + E}{B + D + F} = \frac{E}{F} = \frac{A}{B},$$

therefore A : B :: A + C + E : B + D + F.

PROPOSITION XIII.

THEOREM.—*If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth;* the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A the first have the same ratio to B the second which C the third has to D the fourth, but C the third a greater ratio to D the fourth, than E the fifth has to F the sixth; then also the first A shall have to the second B a greater ratio than the fifth E has to the sixth F.



(*a*) V. Def. 7. (*b*) Hypoth. (*c*) V. Def. 5.

Demonstration. Because C has a greater ratio to D than E to F, there are some equimultiples of C and E, and some of D and F, such that the multiple of C is greater than the multiple of D, but the multiple of E is not greater than the multiple of F (*a*): let these be taken, and let G, H be equimultiples of C, E, and K, L equimultiples of D, F, such that G may be greater than K, but H not greater than L: and whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B: then, because A is to B as C is to D (*b*), and of A and C, M and G are equimultiples; and of B and D, N and K are equimultiples; if M be greater than N, G is greater than K; and if equal, equal; if less, less (*c*): but G is greater than K; therefore M is greater than N: but H is not greater than L: and M, H are equimultiples of A, E; and N, L equimultiples of B, F; *therefore* A *has a greater ratio to* B *than* E *has to* F (*a*).

Corollary. And if the first has a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

Scholium. This proposition may be algebraically expressed as follows:—

Theorem. *If* A : B :: C : D, *but* C : D *is* > E : F; then A : B is E : F.
For C : D is > E : F,

and A : B = C : D,

therefore A : B is > E : F.

PROPOSITION XIV.

THEOREM.—*If the first has to the second the same ratio which the third has to the fourth; then, if the first be greater than the third,* the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let the first A, have to the second B, the same ratio which the third C has to the fourth D; then if A be greater than C, B is greater than D.

Fig. 1.

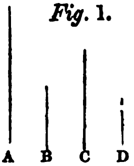


Fig. 2.

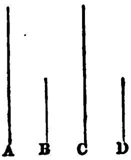
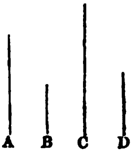


Fig. 3.



(*a*) V. 8.
(*b*) Hypoth.

(*c*) V. 13.
(*d*) V. 10.

(*e*) V. 9.

DEMONSTRATION. Because A is greater than C, and B is any other magnitude, A has to B a greater ratio than C has to B (*a*) **: but, as A is to B, so is C to D** (*b*) **; therefore also C has to D a greater ratio than C has to B** (*c*) **: but of two magnitudes, that to which the same has the greater ratio is the lesser** (*d*) **: therefore D is less than B; that is, B** ***is greater than*** **D.**

Secondly, if A be equal to C, B is equal to D. For A is to B, as C, that is A, is to D; B ***therefore is equal to*** **D** (*e*).

Thirdly, if A be less than C, B is less than D. For C is greater than A; and because C is to D as A is to B, therefore D is greater than B, by the first case; that is, B ***is less than*** **D.**

SCHOLIUM. The foregoing proposition may be expressed algebraically as follows :—

THEOREM. *If* A : B :: C : D, *then if* A *be* $>$ C, B is $>$ D; *if* A $=$ C, B $=$ D; *and if* A *be* $<$ C, B is $<$ D.

For because A : B :: C : D

$$\frac{A}{B}=\frac{C}{D};$$

$$\text{therefore } \frac{A}{C} = \frac{B}{D},$$

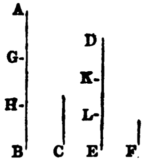
$$\text{and therefore } A : C :: B : D;$$

whence by the 5th Definition it follows that if A be > C, B is > D · if equal, equal; and if less, less.

PROPOSITION XV.

THEOREM.—Magnitudes have the same ratio to one another which their *equimultiples* have.

Let AB be the same multiple of C, that DE is of F; then C is to F, as AB is to DE.



(a) V. 7.
(b) V. 12.

DEMONSTRATION. Because AB is the same multiple of C that DE is of F, there are as many magnitudes in AB equal to C as there are in DE equal to F: let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE: then the number of the first AG, GH, HB is equal to the number of the last DK, KL, LE: and because AG, GH, HB are all equal, and that DK, KL, LE are also equal to one another; therefore AG is to DK, as GH is to KL, and as HB is to LE (*a*): but as one of the antecedents to its consequent, so are all the antecedents together to all the consequents together (*b*); wherefore, as AG is to DK, so is AB to DE: but AG is equal to C, and DK to F; *therefore* C *is to* F, *as* AB *is to* DE.

SCHOLIUM. This proposition may be algebraically expressed as follows:—

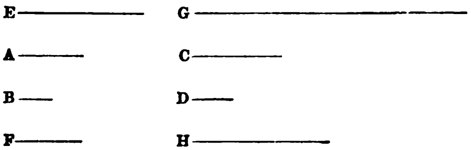
THEOREM. $A : B :: m \,.\, A : m \,.\, B$.

$$\text{For } \frac{A}{B} = \frac{m \,.\, A}{m \,.\, B}.$$

PROPOSITION XVI.

THEOREM.—*If four magnitudes of the same kind be proportionals*, they are also proportionals when taken alternately.

Let A, B, C, D be four magnitudes of the same kind, and let A be to B as C is to D: they are also proportionals when taken alternately; that is, A is to C, as B is to D.



(a) V. 15.
(b) Hypoth.
(c) V. 11.
(d) V. 14.
(e) V. Def. 5.

Demonstration. Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H; and because E is the same multiple of A that F is of B, and that magnitudes have the same ratio to one another which their equimultiples have (*a*); therefore A is to B, as E is to F: but A is to B as C is to D (*b*); wherefore C is to D, as E is to F (*c*): again, because G, H are equimultiples of C, D, therefore C is to D, as G is to H (*a*): but it was proved that C is to D as E is to F; therefore E is to F as G is to H (*c*). But when four magnitudes are proportionals, if the first be greater than the third, the second is greater than the fourth; and if equal, equal: if less, less (*d*): therefore if E be greater than G, F likewise is greater than H; and if equal, equal; if less, less: and E, F are any equimultiples whatever of A, B; and G, H any whatever of C, D: *therefore* A *is to* C, *as* B *is to* D (*e*).

Scholium. It is necessary that the four magnitudes should be of the *same kind* because otherwise a ratio would be established between heterogeneous quantities.

This proposition may be algebraically expressed as follows:—

Theorem. *If* $A : B :: C : D$; then $A : C :: B : D$.

For if $A : B :: C : D$,

$$\frac{A}{B} = \frac{C}{D};$$

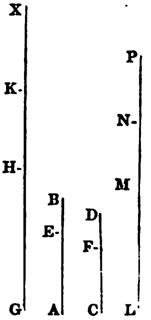
$$\text{therefore } \frac{A}{C} = \frac{B}{D},$$

and therefore $A : C :: B : D$.

PROPOSITION XVII.

Theorem.—*If magnitudes, taken jointly, be proportionals,* they shall also be proportionals when taken separately; that is, *if two magnitudes together have to one of them the same ratio which two others have to one of these,* the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let AB, BE, CD, DF be the magnitudes taken jointly which are proportionals; that is, as AB is to BE so is CD to DF; they shall also be proportionals taken separately, viz. as AE is to EB so is CF to FD.



(a) V. 1.
(b) V. 2.
(c) Hypoth.
(d) V. Def. 5.

Demonstration. Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD take any equimultiples whatever KX, NP: and because GH is the same multiple of AE that HK is of EB, therefore GH is the same multiple of AE that GK is of AB (*a*); but GH is the same multiple of AE that LM is of CF; therefore GK is the same multiple of AB that LM is of CF. Again, because LM is the same multiple of CF that MN is of FD; therefore LM is the same multiple of CF that LN is of CD (*a*): but LM was shown to be the same multiple of CF that GK is of AB; therefore GK is the same multiple of AB that LN is of CD; that is, GK, LN are equimultiples of AB, CD. Next, because HK is the same multiple of EB that MN is of FD; and that KX is also the same multiple of EB that NP is of FD; therefore HX is the same multiple of EB that MP is of FD (*b*). And because AB is to BE as CD is to DF (*c*), and that of AB and CD, GK and LN are equimultiples, and of EB and FD, HX and MP are equimultiples; therefore if GK be greater than HX, then LN is greater than MP; and if equal, equal; if less, less (*d*): but if GH be greater than KX, then, by adding the common part HK to both, GK is greater than HX; wherefore also LN is greater than MP; and by taking away MN from both, LM is greater than NP: therefore if GH be greater than KX, LM is greater than NP. In like manner it may be demonstrated, that if GH be equal to KX, LM is equal to NP; and if less, less: but GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD: *therefore as* AE *is to* EB *so is* CF *to* FD (*d*).

Scholium. This proposition, algebraically expressed, is as follows:—

Theorem. *If* $A + B : B :: C + D : D$; then $A : B :: C : D$.
Because $A + B : B :: C + D : D$,

$$\text{therefore } \frac{A + B}{B} = \frac{C + D}{D};$$

$$\text{or } \frac{A}{B} + 1 = \frac{C}{D} + 1,$$

$$\text{therefore } \frac{A}{B} = \frac{C}{D},$$

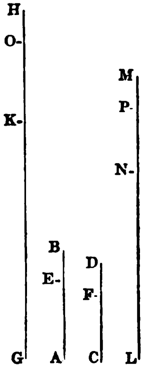
and therefore A : B :: C : D.

PROPOSITION XVIII.

THEOREM.—*If magnitudes, taken separately, be proportionals,* they shall also be proportionals when taken jointly; that is, *if the first be to the second as the third to the fourth,* the first and second together shall be to the second as the third and fourth together to the fourth.

Let AE, EB, CF, FD be proportionals; that is, as AE is to EB, so is CF to FD; they shall also be proportionals when taken jointly; that is, as AB is to BE, so shall CD be to DF.

DEMONSTRATION. Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN; and again, of BE, DF, take any equimultiples whatever of KO, NP: and because KO, NP are equimultiples of BE, DF, and that KH, NM are likewise equimultiples of BE, DF; therefore if KO, the multiple of BE, be greater than KH, which is a multiple of the same BE, then NP, the multiple of DF, is also greater than NM, the multiple of the same DF; and if KO be equal to KH, NP is equal to NM; and if less, less.

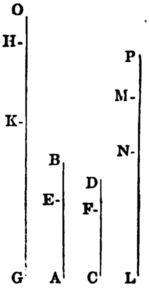


First, let KO be not greater than KH; therefore NP is not greater than NM: and because GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is greater than HK (*a*); but KO is not greater than KH; therefore GH is greater than KO. In like manner, it may be shown that LM is greater than NP. Therefore if KO be not greater than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewise LM, the multiple of CD, is greater than NP, the multiple of DF.

Next, let KO be greater than KH; therefore, as has been shown, NP is greater than NM: and because the whole GH is the same multiple of the whole AB that HK is of BE, therefore the remainder GK is the same mul-

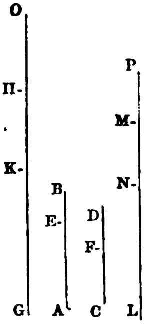
(*a*) V. Ax. 3.

tiple of the remainder AE that GH is of AB (*b*); which is the same that LM is of CD. In like manner, because LM is the same multiple of CD that MN is of DF, therefore the remainder LN is the same multiple of the remainder CF that the whole LM is of the whole CD (*b*): but it was shown that LM is the same multiple of CD that GK is of AE; therefore GK is the same multiple of AE that LN is of CF; that is, GK, LN are equimultiples of AE, CF. And because KO, NP are equimultiples of BE, DF, therefore if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF, the remainders HO, MP are either equal to BE, DF, or equimultiples of them (*c*). First, let HO, MP be equal to BE, DF: then because AE is to EB as CF is to FD (*d*), and that GK, LN are equimultiples of AE, CF; therefore GK is to EB as LN is to FD (*e*): but HO is equal to EB, and MP to FD; wherefore GK is to HO as LN is to MP: therefore if GK be greater than HO, LN is greater than MP (*f*); and if equal, equal; and if less, less.



(*b*) V. 5.
(*c*) V. 6.
(*d*) Hypoth.
(*e*) V. 4, cor.
(*f*) V. Ax. 5.

But let HO, MP be equimultiples of EB, FD: then because AE is to EB as CF is to FD (*d*), and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN is greater than MP; and if equal, equal; if less, less (*g*); which was likewise shown in the preceding case. But if GH be greater than KO, taking KH from both, GK is greater than HO; wherefore also LN is greater than MP; and consequently adding NM to both, LM is greater than NP: therefore if GH be greater than KO, LM is greater than NP. In like manner it may be shown, that if GH be equal to KO, LM is equal to NP; and if less, less. And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and likewise LM greater than NP: but GH, LM are any equimultiples whatever of AB, CD, and KO, NP are any whatever of BE, DF; *therefore as* AB *is to* BE *so is* CD *to* DF (*g*).



(*g*) V. Def. 5.

SCHOLIUM. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If* A : B :: C : D, then A + B : B :: C + D : D.

Because A : B :: C : D,

$$\frac{A}{B} = \frac{C}{D};$$

$$\text{therefore } \frac{A}{B} + 1 = \frac{C}{D} + 1,$$

$$\text{or } \frac{A + B}{B} = \frac{\qquad}{D}$$

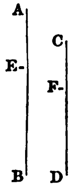
and therefore A + B : B :: C + D : D.

PROPOSITION XIX.

Theorem.—*If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other;* the remainder shall be to the remainder, as the whole to the whole.

Let the whole AB, be to the whole CD, as AE, a magnitude taken from AB, to CF, a magnitude taken from CD; then the remainder EB, shall be to the remainder FD, as the whole AB to the whole CD.

Demonstration. Because AB is to CD, as AE is to CF; therefore alternately, as AB is to AE, so is CD to CF (*a*): and because if magnitudes taken jointly be proportionals, they are also proportionals when taken separately (*b*); therefore EB is to AE, as FD is to CF; and alternately, EB is to FD, as AE is to CF; but AE is to CF, as AB is to CD (*c*); *therefore also the remainder* EB, *is to the remainder* FD, *as the whole* AB *is to the whole* CD (*d*).



(*a*) V. 16.
(*b*) V. 17.
(*c*) Hypoth.
(*d*) V. 11.

Corollary. If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall likewise be to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

Scholium. The foregoing proposition, algebraically expressed, is as follows:—

Theorem. *If* A + B : C + D :: B : D; then A : C :: A + B : C + D.

Because $A + B : C + D :: B : D$,

$$\frac{A+B}{C+D} = \frac{B}{D},$$

$$\text{and } \frac{A+B}{B} = \frac{C+D}{D};$$

$$\text{therefore } \frac{A}{B} + 1 = \frac{C}{D} + 1,$$

$$\text{or } \frac{A}{B} = \frac{C}{D},$$

$$\text{and } \frac{A}{C} = \frac{B}{D};$$

$$\text{therefore } \frac{A}{C} = \frac{A+B}{C+D},$$

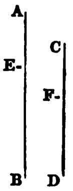
and therefore $A : C :: A + B : C + D$.

PROPOSITION E.

THEOREM.—*If four magnitudes be proportionals*, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB be to BE, as CD is to DF; then AB is to AE, as CD is to CF.

DEMONSTRATION. Because AB is to BE, as CD is to DF, by *division* (*a*), AE is to BE, as CF is to DF; and by inversion (*b*), BE is to AE, as FD is to CF. Wherefore by *composition* (*c*), AB *is to* AE, *as* CD *is to* CF.



(*a*) V. 17.
(*b*) V. B.
(*c*) V. 18.

SCHOLIUM. This proposition has been added by Simson. the meaning of the terms "by division," and "by composition," are those explained in the 17th and 16th definitions. The foregoing proposition may be algebraically expressed as follows:—

THEOREM. *If* $A : B :: C : D$; then $A : A \sim B :: C : C \sim D$.

Because $A : B :: C : D$,

$$\frac{A}{B} = \frac{C}{D};$$

$$\text{therefore } \frac{A}{B} \sim 1 = \frac{C}{D} \sim 1,$$

$$\text{and } \frac{A \sim B}{B} = \frac{C \sim D}{D};$$

$$\text{therefore } \frac{A \sim B}{C \sim D} = \frac{B}{D};$$

$$\text{but } \frac{A}{C} = \frac{B}{D},$$

$$\text{therefore } \frac{A \sim B}{C \sim D} = \frac{A}{C},$$

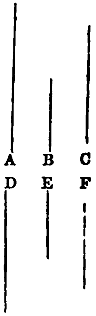
$$\text{and } \frac{A}{A \sim B} = \frac{C}{C \sim D},$$

therefore A : A ~ B :: C : C ~ D.

PROPOSITION XX.

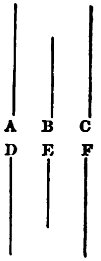
THEOREM.—*If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third,* the fourth shall be greater than the sixth; *and if equal,* equal; *and if less,* less.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, viz. as A is to B, so is D to E; and as B is to C, so is E to F; then if A be greater than C, D shall be greater than F; and if equal, equal; if less, less.



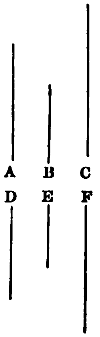
DEMONSTRATION. First, let A be greater than C; D shall be greater than F. For because A is greater than C, and B is any other magnitude, and that the greater has to the same magnitude a greater ratio than the less has to it (*a*); therefore A has to B a greater ratio than C has to B: but as D is to E, so is A to B (*b*); therefore D has to E a greater ratio than C has to B (*c*); and because B is to C, as E is to F, by inversion, C is to B as F is to E; and D was shown to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E (*d*); but the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two (*e*); *therefore D is greater than F.*

(*a*) V. 8.
(*b*) Hypoth.
(*c*) V. 13.
(*d*) V. 13, cor
(*e*) V. 10.



Secondly, let A be equal to C; then D shall be equal to F. Because A and C are equal to one another, A is to B as C is to B (*f*) : but A is to B as D is to E; and C is to B as F is to E; wherefore D is to E as F is to E (*g*); and *therefore* D *is equal to* F (*h*).

(*f*) V. 7.
(*g*) V. 11.
(*h*) V. 9.



Thirdly, let A be less than C; then D shall be less than F: for C is greater than A, and, as was shown in the first case, C is to B, as F is to E, and in like manner B is to A, as E is to D; therefore F is greater than D, by the first case; and *therefore* D *is less than* F.

Scholium. The foregoing proposition may be algebraically expressed as follows:—

Theorem. *If* A, B, C *be three magnitudes, and* D, E, F *three others, and if* A : B :: D : E, *and* B : C :: E : F; *then, if* A *be* > C, D is also > F; *and if equal*, equal; *if less*, less.

Because A : B :: D : E,

$$\frac{A}{B} = \frac{D}{E};$$

and because B : C :: E : F,

$$\frac{B}{C} = \frac{E}{F};$$

$$\text{therefore } \frac{A}{B} \cdot \frac{B}{C} = \frac{D}{E} \cdot \frac{E}{F},$$

$$\text{or, } \frac{A}{C} = \frac{D}{F};$$

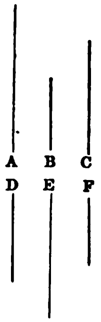
therefore A : C :: D : F,

whence by the 5th definition it follows that if A is > C, D is > F; and if equal, equal; and if less, less.

PROPOSITION XXI.

THEOREM.—*If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; then if the first magnitude be greater than the third,* the fourth shall be greater than the sixth; *and if equal,* equal; *and if less,* less.

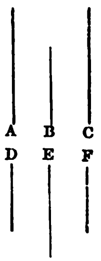
Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order, viz. as A is to B, so is E to F, and as B is to C, so is D to E; then if A be greater than C, D shall be greater than F; and if equal, equal; and if less, less.



DEMONSTRATION. First, let A be greater than C; then D shall be greater than F: for because A is greater than C, and B is any other magnitude, A has to B a greater ratio than C has to B (*a*): but as E is to F, so is A to B (*b*); therefore E has to F a greater ratio than C to B (*c*); and because B is to C, as D is to E (*b*), by inversion, C is to B, as E is to D: and E was shown to have to F a greater ratio than C to B; therefore E has to F a greater ratio than E to D (*d*); but the magnitude to which the same has a greater ratio than it has to another, is the lesser of the two (*e*); therefore F is less than D; that is, D *is greater than* F.

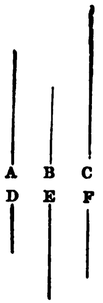
(*a*) V. 8.
(*b*) Hypoth.
(*c*) V. 13.
(*d*) V. 13, cor
(*e*) V. 10.

Secondly, let A be equal to C; then D shall be equal to F. Because A and C are equal, A is to B, as C is to B (*f*): but A is to B, as E to F; and C is to B as E is to D; wherefore E is to F as E is to D (*g*); and *therefore* D *is equal to* F (*h*).



(*f*) V. 7.
(*g*) V. 11.
(*h*) V. 9.

Thirdly, let A be less than C; then D shall be less than F. For C is greater than A, and, as was shown, C is to B, as E is to D, and in like manner, B is to A, as F is to E; therefore F is greater than D, by the first case; and *therefore* D *is less than* E.



SCHOLIUM. This proposition may be algebraically expressed as follows:—

THEOREM. *If* A, B, C *be three magnitudes, and* D, E F *three others, such that* A : B :: E : F, *and* B : C :: D : E; *then if* A *be* > C, D is also > F; *and if equal*, equal; *if less*, less.

Because A : B :: E : F,

$$\frac{A}{B} = \frac{E}{F};$$

and because B : C :: D : E,

$$\frac{B}{C} = \frac{D}{E};$$

$$\text{then } \frac{A}{B} \cdot \frac{B}{C} = \frac{E}{F} \cdot \frac{D}{E};$$

$$\text{or } \frac{A}{C} = \frac{D}{F},$$

therefore A : C :: D : F,

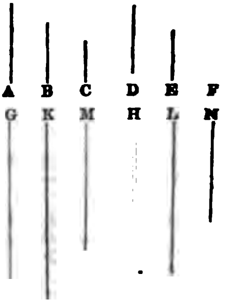
whence by the 5th definition it follows that if A is > C, D is > F; and if equal, equal; if less, less.

PROPOSITION XXII.

THEOREM.—*If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio;* the first has to the last of the first magnitudes the same ratio which the first has to the last of the others.

DEMONSTRATION. First let there be three magnitudes A, B, C, and as many others D, E, F, which, taken two and two, have the same ratio, that is, such that A is to B, as D is to E; and that B is to C, as E is to F; then A shall be to C, as D is to F.

Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N. Then, because A is to B, as D is to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, so is H to L (a): for the same reason, K is to M, as L is to N: and because there are three magnitudes G, K, M, and other three, H, L, N, which, two and two, have the same ratio; if G be greater than M, H is greater than N; and if equal, equal; if less, less (b): and G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F; *therefore, as* A *is to* C, *so is* D *to* F (c).



(a) V. 4.
(b) V. 20.
(c) V. Def. 5.

Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, two and two, have the same ratio, viz. as A is to B, so is E to F; and as B is to C, so is F to G; and as C is to D, so is G to H: then shall A be to D, as E is to H.

A. B. C. D.
E. F. G. H.

Because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio; by the foregoing case, A is to C, as E is to G: but C is to D, as G is to H; wherefore again, by the first case, A *is to* D, *as* E *is to* H: *and so on, whatever be the number of magnitudes.*

SCHOLIUM. This proposition is expressed by the terms, *ex æquali*, or *ex æquo*, as explained in the 20th definition. It may be algebraically expressed as follows:—

THEOREM. *If* A, B, C, D *be any magnitudes, and* E, F, G, H *be as many others, such that*

$$\begin{aligned} A : B &:: E : F, \\ B : C &:: F : G, \\ C : D &:: G : H, \\ &\&c. \end{aligned}$$

then, *ex æquo*, A : D :: E : H.

$$\text{For } \frac{A}{B} = \frac{E}{F}, \quad \frac{B}{C} = \frac{F}{G}, \quad \frac{C}{D} = \frac{G}{H}, \quad \&c.$$

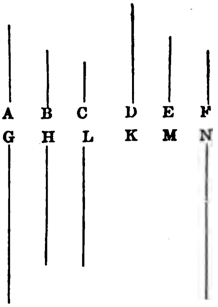
$$\text{Then } \frac{A}{B}\cdot\frac{B}{C}\cdot\frac{C}{D}=\frac{E}{F}\cdot\frac{F}{G}\cdot\frac{G}{H},$$

$$\text{or } \frac{A}{D}=\frac{E}{H},$$

therefore A : D :: E : H.

PROPOSITION XXIII.

THEOREM.—*If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio;* the first has to the last of the first magnitudes the same ratio which the first has to the last of the others.



DEMONSTRATION. First, let there be three magnitudes A, B, C, and other three, D, E, F, which, taken two and two, in a cross order, have the same ratio; that is, such that A is to B, as E is to F; and that B is to C, as D is to E; then shall A be to C, as D is to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N; and because G, H are equimultiples of A, B, and that magnitudes have the same ratio which their equimultiples have (*a*); as A is to B, so is G to H: and for the same reason, as E is to F, so is M to N: but as A is to B, so is E to F; as therefore G is to H, so is M to N (*b*). And because as B is to C, so is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; as H is to L, so is K to M (*c*): and it has been shown that G is to H, as M is to N: then, because there are three magnitudes G, H, L, and other three K, M, N which have the same ratio taken two and two, in a cross order; if G be greater than L, K is greater than N; and if equal, equal; if less, less (*d*); and G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; *therefore as* A *is to* C, *so is* D *to* F (*e*).

(*a*) V. 15.
(*b*) V. 11.
(*c*) V. 4.
(*d*) V. 21.
(*e*) V. Def. 5.

Next, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which, taken two and two, in a cross order, have the same ratio, viz. A is to B, as G is to H; B is to C, as F is to G; and C is to D, as E is to F: then shall A be to D, as E is to H.

A. B. C. D.
E. F. G. H.

Because A, B, C are three magnitudes, and F, G, H other three, which, taken two and two, in a cross order, have the same ratio; by the first case, A is to C, as F is to H: but C is to D, as E is to F; wherefore again, by the first case, A *is to* D, *as* E *is to* H: *and so on, whatever be the number of magnitudes.*

SCHOLIUM. This proposition is expressed by the terms, *ex æquali in proportione perturbata*, or *ex æquo perturbate*, as explained in the 21st definition. Algebraically expressed, it is as follows:—

THEOREM. *If* A, B, C, D *be any magnitudes, and* E, F, G, H *as many others, such that*

$$A : B :: G : H,$$
$$B : C :: F : G,$$
$$C : D :: E : F,$$

then, *ex æquo perturbate,*

$$A : D :: E : H.$$

$$\text{For } \frac{A}{B} = \frac{G}{H},$$

$$\frac{B}{C} = \frac{F}{G},$$

$$\frac{C}{D} = \frac{E}{F},$$

&c.

$$\text{Then } \frac{A}{B} \cdot \frac{B}{C} \cdot \frac{C}{D} = \frac{G}{H} \cdot \frac{F}{G} \cdot \frac{E}{F},$$

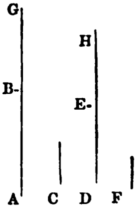
$$\text{or } \frac{A}{D} = \frac{E}{H},$$

therefore A : D :: E : H.

PROPOSITION XXIV.

THEOREM.—*If the first have to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth;* the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB the first, have to C the second, the same ratio which DE the third, has to F the fourth ; and let BG the fifth, have to C the second, the same ratio which EH the sixth, has to F the fourth: then, AG the first and fifth together, shall have to C the second, the same ratio which DH, the third and sixth together, has to F the fourth.



DEMONSTRATION. Because BG is to C, as EH is to F; by inversion, C is to BG, as F is to EH: and because AB is to C, as DE is to F; and C is to BG, as F is to EH; *ex æquali*, AB is to BG, as DE is to EH (*a*): and because these magnitudes are proportionals, they shall likewise be proportionals when taken jointly (*b*); therefore as AG is to BG, so is DH to EH; but as BG is to C, so is EH to F. Therefore, *ex æquali*, as AG is to C, so is DH to F (*a*).

(*a*) V. 22.
(*b*) V. 18.

COROLLARY 1. If the same hypothesis be made as in the proposition, the difference of the first and fifth shall be to the second, as the difference of the third and sixth to the fourth: the demonstration of this is the same with that of the proposition, if "*division*" be used instead of "*composition*."

COROLLARY 2. This proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

SCHOLIUM. The foregoing proposition, algebraically expressed, is as follows:—

THEOREM. *If* A : B :: C : D, *and* E : B :: F : D; then A + E : B :: C + F : D.

$$\text{For } \frac{A}{B} = \frac{C}{D},$$

$$\text{therefore } \frac{A}{C} = \frac{B}{D};$$

$$\text{Also } \frac{E}{B} = \frac{F}{D},$$

$$\text{and } \frac{E}{F} = \frac{B}{D};$$

$$\text{therefore } \frac{A}{C} = \frac{E}{F};$$

$$\text{and } \frac{A}{E}=\frac{C}{F}.$$

Adding unity to each side,

$$\frac{A}{E}+1=\frac{C}{F}+1,$$

$$\text{and } \frac{A+E}{E}=\frac{C+F}{F}.$$

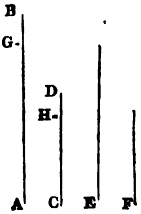
$$\text{therefore } \frac{A+E}{C+F}=\frac{E}{F}=\frac{B}{D},$$

$$\text{therefore } \frac{A+E}{B}=\frac{C+F}{D},$$

and A + E : B :: C + F : D.

PROPOSITION XXV.

THEOREM.—*If four magnitudes of the same kind are proportionals,* the greatest and least of them together are greater than the other two together.



(a) V. A & 14.
(b) V. 19.
(c) V. A.

DEMONSTRATION. Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greatest of them, and consequently F the least (*a*); then shall AB together with F, be greater than CD together with E.

Take AG equal to E, and CH equal to F: then, because as AB is to CD, so is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG is to CH; and because AB the whole, is to the whole CD, as AG is to CH, likewise the remainder GB shall be to the remainder HD, as the whole AB is to the whole CD (*b*): but AB is greater than CD, therefore GB is greater than HD (*c*).; and because AG is equal to E, and CH to F, AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and to HD the two CH and E; AB and F together are greater than CD and E.

SCHOLIUM. This proposition, may be algebraically expressed as follows:—

THEOREM. *If* A : B :: C : D, *and if* A *is the greatest of them,* then A + D is > B + C.

$$\text{For } \frac{A}{B} = \frac{C}{D},$$

multiplying by B,

$$A = \frac{C \cdot B}{D},$$

adding D,

$$A + D = \frac{C \cdot B}{D} + D,$$

and subtracting B + C,

$$\begin{aligned} A + D - (B + C) &= \frac{C \cdot B}{D} + D - (B + C) \\ &= C \cdot \left(\frac{B}{D} - 1\right) + D - B \\ &= C \cdot \frac{B - D}{D} - (B - D) \\ &= \left(\frac{C}{D} - 1\right) \cdot (B - D) \\ &= \frac{(C - D) \cdot (B - D)}{D} \end{aligned}$$

Now by the 5th Definition if A is > B, and also > C, then C is > D, and B is > D, therefore both (C − D) and (B − D) are positive; therefore,

$$A + D \text{ is} > B + C \text{ by } \frac{(C - D) \cdot (B - D)}{D}.$$

PROPOSITION F.

Theorem.—*If ratios are compounded of the same ratios,* they are the same with one another.

Demonstration. Let A be to B, as D is to E; and B to C, as E is to F: then the ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, is the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.

A	B	C
D	E	F

Because there are three magnitudes A, B, C, and three others

D, E, F, which, taken two and two in order, have the same ratio; *ex æquali*, A is to C, as D is to F (*a*).

Next, let A be to B, as E is to F, and B to C, as D is to E; therefore, *ex æquali in proportione perturbatâ*, as A is to C, so is D to F (*b*); that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F; and in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

A B C
D E F

(*a*) V. 22.
(*b*) V. 23.

SCHOLIUM. This and the three following propositions have been added by Simson; the two last, propositions H and K, are not read at the Universities.

The foregoing proposition may be algebraically expressed as follows, and its truth is then evident:—

THEOREM. *If* A : B :: E : F
B : C :: F : G
C : D :: G : H,
&c.

Or if A : B :: G : H
B : C :: F : G
C : D :: E : F,
&c.

then *ex æquo*, or *ex æquo perturbatè*,

A : D :: E : H.

PROPOSITION G.

THEOREM.—*If several ratios be the same with several ratios, each to each;* the ratio which is compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.

A B C D K L M
E F G H N O P

DEMONSTRATION. Let A be to B, as E is to F; and C to D, as G is to H: and let A be to B, as K is to L; and C to D, as L is to M: then the ratio of K to M, by the definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the same with the ratios of A to B, and C to D: and as E is to F, so let N be to O; and as G to H, so let O be to P; then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the same with the ratios of E to F, and G to H: and it is to be shown that the ratio of K to M, is the same with the ratio of N to P, or that K is to M, as N is to P.

Because K is to L, as (A is to B, that is, as E is to F, that is, as) N is to O; and L is to M, as (C is to D, that is, as G is to H, that is, as) O is to P: therefore, *ex æquali*, K is to M, as N is to P (*a*).

A	B	C	D	K	L	M
E	F	G	H	N	O	P

(*a*) V. 22.

SCHOLIUM. This proposition may be algebraically expressed as follows:—

THEOREM. *If* A : B :: E : F
and C : D :: G : H
Also, if A : B :: K : L
C : D :: L : M
E : F :: N : O
and G : H :: O : P
then K : M :: N : P

$$\text{For } \frac{A}{B} = \frac{K}{L}, \text{ and } \frac{C}{D} = \frac{L}{M},$$

$$\text{therefore } \frac{A}{B} \cdot \frac{C}{D} = \frac{K}{L} \cdot \frac{L}{M} = \frac{K}{M};$$

$$\text{and } \frac{E}{F} = \frac{N}{O}, \text{ and } \frac{G}{H} = \frac{O}{P},$$

$$\text{therefore } \frac{E}{F} \cdot \frac{G}{H} = \frac{N}{O} \cdot \frac{O}{P} = \frac{N}{P};$$

$$\text{Again } \frac{A}{B} = \frac{E}{F}, \text{ and } \frac{C}{D} = \frac{G}{H},$$

$$\text{therefore } \frac{A}{B} \cdot \frac{C}{D} = \frac{E}{F} \cdot \frac{G}{H};$$

$$\text{therefore } \frac{K}{M} = \frac{N}{P},$$

and K : M :: N : P.

PROPOSITION H.

THEOREM.—*If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first,* *be the same with one of the last ratios, or with the ratio compounded of any of the last;* then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.

Demonstration. Let the first ratios be those of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be those of G to H, H to K, K to L, and L to M: also, let the ratio of A to F, which is compounded of the first ratios, be the same with the ratio of G to M, which is compounded of the other ratios: and besides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, and C to D, be the same with the ratio of G to K, which is compounded of the ratios of G to H, and H to K: then the ratio compounded of the remaining first ratios, viz. of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, is the same with the ratio of K to M. which is compounded of the remaining ratios of K to L, and L to M, of the other ratios.

A	B	C	D	E	F
G	H	K	L	M	

Because, by the hypothesis, A is to D, as G is to K; by inversion, D is to A, as K is to G (*a*); and as A is to F, so is G to M; therefore, *ex æquali*, D is to F, as K is to M (*b*).

(*a*) V. B.
(*b*) V. 22.

Scholium. The foregoing proposition, algebraically expressed, is as follows:—

Theorem. *If* A : F *be compounded of* A : B, B : C, C : D, D : E, E : F, *and* G : M, *be compounded of* G : H, H : K, K : L, L : M;

$$\textit{and if } A : F :: G : M,$$
$$\textit{and } A : D :: G : K,$$
$$\text{then } D : F :: K : M,$$

$$\text{For } \frac{A}{D} = \frac{G}{K},$$

$$\text{therefore } \frac{A}{G} = \frac{D}{K},$$

$$\text{and } \frac{D}{A} = \frac{K}{G},$$

$$\text{also } \frac{A}{F} = \frac{G}{M},$$

$$\text{Then } \frac{D}{A} \cdot \frac{A}{F} = \frac{K}{G} \cdot \frac{G}{M},$$

$$\text{or } \frac{D}{F} = \frac{K}{M},$$

therefore D : E :: K : M.

PROPOSITION K.

Theorem.—*If there be any number of ratios, and any number of other ratios such that the ratio compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same, each to each, with the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same with several of the first ratios, each to each, be the same with one of the last ratios, or with the ratio compounded of ratios which are the same, each to each, with several of the last ratios:* then the ratio compounded of ratios which are the same with the remaining ratios of the first, each to each, or the remaining ratio of the first, if but one remain; is the same with the ratio compounded of ratios which are the same with those remaining of the last, each to each, or with the remaining ratio of the last.

Demonstration. Let the ratios of A to B, C to D, E to F be the first ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: and let A be to B, as S is to T; and C to D, as T is to V; and E to F, as V is to X; therefore, by the definition of compound ratio, the ratio S to X is compounded of the ratios of S to T, T to V, and V to X, which are the same with the ratios of A to B, C to D, E to F, each to each; also, let G be to H, as Y is to Z; and K to L, as Z is to a; M to N, as a is to b; O to P, as b is to c; and Q to R, as c is to d: therefore, by the same definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and c to d, which are the same,

h, k, l.
A, B; C, D; E, F; S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d,
e, f, g. m, n, o, p.

each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R: therefore, by the hypothesis, S is to X, as Y is to D: also, let the ratio of A to B, that is, the ratio of S to T, which is one of the first ratios, be the same with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypothesis, are the same with the ratios of G to H, and K to L, two of the other ratios; and let the other ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the same with the remaining first ratios, viz. of C to D, and E to F; also, let the ratio of m to p be that which is compounded of the ratios of m to n, n to o, and o to p, which are the same, each to each, with the remaining other ratios, viz. of M to N, O to P, and Q to R: then the ratio of h to l is the same with the ratio of m to p; that is, h is to l, as m is to p.

h, k, l.
A, B; C, D; E, F; S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d,
e, f, g. m, n, o, p.

(*a*) V. 11.

Because e is to f, as (G is to H, that is, as) Y is to Z; and f is to g, as (K is to L, that is, as) Z is to a; therefore, *ex æquali*, e is to g, as Y is to a: and by the hypothesis, A is to B, that is, S is to T, as e is to g; wherefore S is to T, as Y is to a; and, by inversion, T is to S, as a is to Y; and S is to X, as Y is to D; therefore, *ex æquali*, T is to X, as a is to d: also, because h is to k, as (C is to D, that is, as) T is to V; and k is to l, as (E is to F, that is, as) V is to X; therefore, *ex æquali*, h is to l, as T is to X: in like manner it may be demonstrated, that m is to p, as a is to d: and it has been shown, that T is to X, as a is to d; therefore h is to l, as m is to p (*a*).

The propositions G and K are usually, for the sake of brevity, expressed in the same terms with propositions F and H; and therefore it was proper to show the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers.

SCHOLIUM. This proposition may be algebraically expressed:—

THEOREM.—*If there be a number of ratios* A : B, C : D, E : F, *and if*

A : B :: S : T
C : D :: T : V :: *h* : *k*
E : F :: V : X :: *k* : *l*

and also a number of other ratios G : H, K : L, M : N, O : P, Q : R, *and if*

$$\begin{array}{l} G : H :: Y : Z :: e : f \\ K : L :: Z : a :: f : g \\ M : N :: a : b :: m : n \\ O : P :: b : c :: n : o \\ Q : R :: c : d :: o : p \end{array}$$

and if $S : X :: Y : d$;

and $A : B :: e : g$; then shall
$h : l :: m : p$.

$$\text{For } \frac{A}{B} = \frac{e}{g},$$

$$\text{But } \frac{A}{B} = \frac{S}{T}, \text{ and } \frac{e}{g} = \frac{Y}{a},$$

$$\text{therefore } \frac{S}{T} = \frac{Y}{a},$$

$$\text{and } \frac{S}{Y} = \frac{T}{a}.$$

$$\text{Then } \frac{T}{X} = \frac{h}{l}:$$

$$\frac{S}{X} = \frac{Y}{d}$$

$$\text{and } \frac{S}{Y} = \frac{X}{d},$$

$$\text{therefore } \frac{T}{a} = \frac{X}{d},$$

$$\text{and } \frac{T}{X} = \frac{a}{d} = \frac{m}{p},$$

$$\text{therefore } \frac{m}{p} = \frac{h}{l},$$

and $m : p .. h : l$.

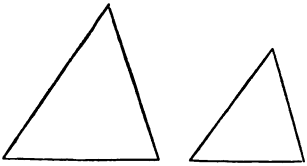
THE

ELEMENTS OF EUCLID.

BOOK VI.

DEFINITIONS.

1. *Similar rectilineal figures* are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

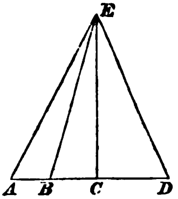


Scholium. In the case of triangles it would have been sufficient to state that 'similar triangles are those which have two of their angles equal,' because it is evident from I, 32 B, the third sides must also be equal, and it is shown in the fourth proposition of this book that the sides about the equal angles of equiangular triangles are proportionals. But in the case of rectilineal figures having more than three sides both the conditions expressed above are necessary, because, as in the case of a square and rectangle, the angles are equal, each to each, but the sides about the equal angles are not proportional.

2. Two magnitudes are said to be *reciprocally proportional* to two others, when one of the first pair is to one of the second, as the remaining one of the second is to the remaining one of the first.

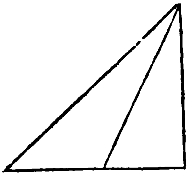
3. A straight line is said to be cut in *extreme and mean ratio*, when the whole is to one of the segments, as that segment is to the other.

Scholium. A straight line is said to be divided *harmonically*, when it is divided into three parts, such that the whole line is to one of the extreme segments, as the other extreme segment is to the middle part. Three lines are said to be in *harmonical proportion*, when the first (AB) is to the third (CD), as the difference between the first (AB) and second (BC), is to the difference between the second (BC) and third (CD); and the second (BC) is called a *harmonic mean* between the first (AB) and third (CD). Four divergent lines (EA, EB, EC, ED) which cut a line (AD) in harmonical proportion, are called *harmonicals;* and this mode of dividing a line is termed *harmonical section*, while that described in the third definition is termed *medial section*.



4. The altitude of any figure is the straight line drawn from its vertex perpendicular to its base.

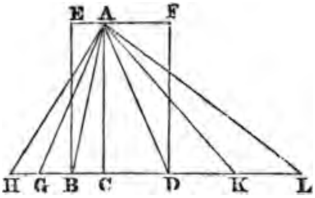
Scholium. Any side of a figure may be assumed as its base, and its altitude is the perpendicular distance from such side to the most remote point in the figure.



PROPOSITION I.

Theorem.—*Triangles* (ABC, ACD) *and parallelograms* (EC, CF) *which have the same altitude,* are to one another as their bases.

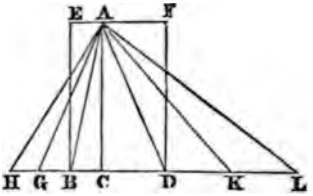
Construction. *Produce* BD *both ways to the points* H, L, *and take any number of straight lines* BG, GH, *each equal to the base* BC; *and* DK, KL, *any number of them, each equal to the base* CD; *and join* AG, AH, AK, AL.



Demonstration. Then because CB, BG, GH are all equal, the triangles ABC, AGB, AHG are all equal (*a*); therefore whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC: for the same reason, whatever multiple the base CL is of the base CD, the same multiple is the triangle ALC of the triangle ADC: and if the base HC be equal to the base CL, the triangle AHC is also equal to the triangle ALC (*a*);

(*a*) I. 38.

and if the base HC be greater than the base CL, likewise the triangle AHC is greater than the triangle ALC; and if less, less: therefore, since there are four magnitudes, viz. the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC, and the triangle ABC, the first and third, any equimultiples whatever have been taken, viz. the base HC, and the triangle AHC; and also of the base CD and the triangle ACD, the second and fourth, any equimultiples whatever have been taken, viz. the base CL, and the triangle ALC; and since it has been shown, that if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less; *therefore, as the base* BC *is to the base* CD, *so is the triangle* ABC *to the triangle* ACD (*b*).



(*b*) V. Def. 5.
(*c*) I. 41.
(*d*) V. 15.
(*e*) V. 11.

And because the parallelogram CE is double of the triangle ABC (*c*), and the parallelogram CF double of the triangle ACD (*c*), and that magnitudes have the same ratio which their equimultiples have (*d*); as the triangle ABC is to the triangle ACD, so is the parallelogram CE to the parallelogram CF: and because it has been shown, that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, so is the parallelogram CE to the parallelogram CF; *therefore, as the base* BC *is to the base* CD, *so is the parallelogram* CE *to the parallelogram* CF (*e*).

COROLLARY 1. From this it is evident, that *triangles and parallelograms which have equal altitudes*, are to one another as their bases.

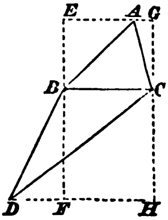
For, let the figures be so placed as to have their bases in the same straight line, and draw perpendiculars from the vertices of the triangles to the bases, then, because the perpendiculars are both equal and parallel to one another (*a*), the straight line which joins the vertices is parallel to that in which their bases are (*b*). Then, if the same construction be made as in the proposition, the demonstration will be identical.

(*a*) I. 28.
(*b*) I. 33.

COROLLARY 2. THEOREM. *Triangles* (ABC, DBC) *and parallelograms which have equal bases*, are to one another as their altitudes.

Construction. *Let* DBC *be so placed that its base shall coincide with that of* ABC, *but their vertices shall be on opposite sides; through the vertices draw* EG *and* DH *parallel to* BC, *and through* B *and* C *draw* EF *and* GH *perpendicular to* BC.

Demonstration. Then because the parallelograms GB and BH have the same altitude BC, as the parallelogram GB is to the parallelogram BH, so is the base GC to the base CH (*a*); but the parallelogram BG is double of the triangle ABC, and the parallelogram BH is double of the triangle DBC, and magnitudes have the same ratio which their equimultiples have; *therefore, the triangle* ABC *is to the triangle* DBC *as the altitude* GC *is to the altitude* CH.



(*a*) VI. 1.

Corollary 3. Theorem. *If neither the bases nor altitudes of triangles and parallelograms are equal,* they are to one another in the compound ratio of their bases and altitudes.

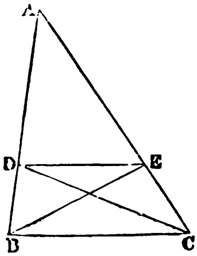
PROPOSITION II.

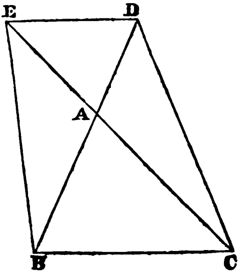
Theorem [1.]—*If a straight line* (DE) *be parallel to the base* (BC) *of a triangle* (ABC), it cuts the other sides, or those sides produced, so that their segments between the parallel and the base (BD and CE) have the same ratio to their segments between the parallel and the vertex (DA, EA); that is, BD is to DA, as CE is to EA.

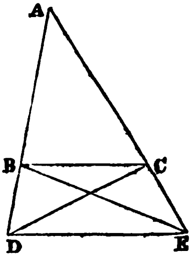
[2.]—*In a triangle* (ABC) *if the sides, or sides produced, be cut by a straight line* (DE), *so that their segments between the straight line and the base* (BD, CE) *have the same ratio as their segments between the straight line and the vertex* (DA, EA); the straight line is parallel to the base.

Construction. *Join* BE, CD.

Demonstration. [1.] The triangles BDE and CDE are equal, because they are on the same base DE, and between the same





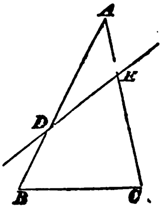


(*a*) I. 37.
(*b*) V. 7.
(*c*) VI. 1.
(*d*) V. 11.
(*e*) Hypoth.
(*f*) V. 9.
(*g*) I. 39.

parallels DE and BC (*a*). Now, ADE is another triangle, and equal magnitudes have to the same the same ratio (*b*), therefore the triangle BDE is to the triangle ADE, as the triangle CDE is to the triangle ADE: but as the triangle BDE is to the triangle ADE, so is BD to DA, because, having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases (*c*); and for the same reason, as the triangle CDE is to the triangle ADE, so is CE to EA: *therefore, as* BD *is to* DA, *so is* CE *to* EA (*d*).

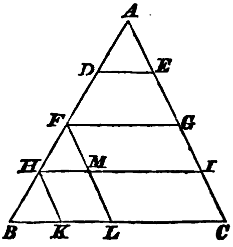
[2.] Because BD is to DA, as CE is to EA (*e*); and BD is to DA, as the triangle BDE is to the triangle ADE (*c*); and CE is to EA, as the triangle CDE is to the triangle ADE (*c*): therefore, the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE (*d*); that is, the triangles BDE, CDE have the same ratio to the triangle ADE; therefore the triangle BDE is equal to the triangle CDE (*f*); and they are on the same base: but equal triangles on the same base are between the same parallels (*g*), *therefore* DE *is parallel to* BC.

SCHOLIUM. This proposition consists of two distinct theorems, each the converse of the other. The enunciation of this proposition, as given by Simson, is very defective, inasmuch as he omits to state which of the segments of the sides are homologous to one another in the proportion: and that of the converse theorem is, strictly speaking, false, since a straight line may cut the sides of a triangle proportionally, without being parallel to the base; as in the figure, where AD is to DB, as CE is to EA. The necessity for three figures in the foregoing proposition arises from the varying position which the line DE may have in reference to the triangle, viz. beyond either the vertex or the base, or between them.



Corollary. **Theorem.** *If several parallels* (DE, FG, HI) *be drawn to the base of a triangle* (ABC), every pair of corresponding segments in each side will be proportional; that is, as AD is to DF, so is AE to EG, and as FH is to HB, so is GI to IC.

Demonstration. For, draw HK and FL parallel to AC. Then in the parallelograms FI, MC, the opposite sides are equal (*a*), therefore, FM equal GI, and ML equal IC; and in the triangles AFG and FBL, AD *is to* DF, *as* AE *is to* EG (*b*); and FH *is to* HB, *as* FM *is to* ML (*b*), that is, *as* GI *is to* IC.



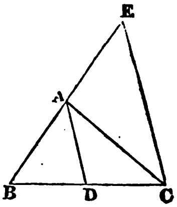
(*a*) I. 34.
(*b*) VI. 2.

PROPOSITION III.

Theorem [1.]—*If the angle of a triangle* (ABC) *be bisected by a straight line* (AD) *which also cuts the base*, the segments of the base (BD, DC) shall have the same ratio which the other sides of the triangle (AB, AC) have to one another.

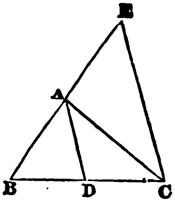
[2.] *And if a straight line* (AD) *drawn from any angle of a triangle* (ABC) *divide the opposite side into segments* (BD, DC) *which have the same ratio as the adjacent sides* (AB, AC), it bisects the angle.

Construction. *Through* C *draw* CE *parallel to* DA (*a*) ; *then* BA *produced will meet* CE (*b*), *let them meet in* E.



Demonstration. Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (*c*): but CAD, by the hypothesis, is equal to the angle BAD: therefore the angle BAD is equal to the angle ACE (*d*). Again, because the straight line BAE meets the parallels AD, EC, the external angle BAD is equal to the internal and opposite angle AEC (*c*): but the angle ACE has been proved equal to the angle BAD ; therefore also ACE is equal to the angle AEC (*d*),

(*a*) I. 31.
(*b*) Theor. attached to I. 29.
(*c*) I. 29.
(*d*) I. Ax. 1.



and consequently the side AE is equal to the side AC (*e*) : and because AD is drawn parallel to EC, one of the sides of the triangle BCE ; therefore BD is to DC, as AB is to AE (*f*): but AE is equal to AC ; *therefore* BD *is to* DC, *as* AB *is to* AC.

[2.] Because AD is parallel to EC, BD is to DC, as AB is to AE (*f*) ; and BD is to DC, as AB is to AC (*g*) ; therefore AB is to AC, as AB is to AE (*h*) ; consequently AC is equal to AE (*i*), and therefore the angle AEC is equal to the angle ACE (*k*); but the angle AEC is equal to the external and opposite angle BAD ; and the angle ACE is equal to the alternate angle CAD (*c*) ; wherefore also the angle BAD is equal to the angle CAD (*d*) ; that is, *the angle* BAC *is bisected by the straight line* AD.

(*e*) I. 6.
(*f*) VI. 2.
(*g*) Hypoth.
(*h*) V. 11.
(*i*) V. 9.
(*k*) I. 5.

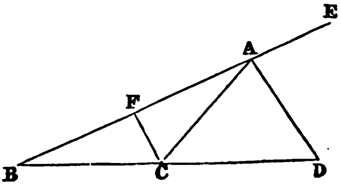
COROLLARY. From this proposition it follows that if the same straight line bisects the angle and the base the triangle is isosceles.

PROPOSITION A.

THEOREM [1.]—*If an exterior angle of a triangle* (ABC) *be bisected by a straight line* (AD) *which also cuts the base produced,* the segments between the bisecting line and the extremities of the base (DB, DC), have the same ratio to one another, as the other sides of the triangle (AB, AC) have.

[2.] *And if the segments* (BD, DC) *of the base produced, have the same ratio which the other sides of the triangle* (AB, AC) *have,* the straight line (AD) drawn from the vertex to the point of section bisects the exterior angle (CAE) of the triangle.

DEMONSTRATION [1.] Let the exterior angle CAE of any triangle ABC, be bisected by the straight line AD which meets the base produced in D ; then BD is to DC, as BA is to AC.



(a) I. 31.
(b) I. 29.
(c) Hypoth.
(d) I. 6.
(e) VI. 2.
(f) V. 11.
(g) V. 9
(h) V. 1.

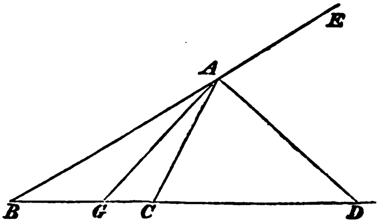
For, *through* C *draw* CF *parallel to* AD (*a*); and because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (*b*): but CAD is equal to the angle DAE (*c*); therefore also DAE is equal to the angle ACF. Again, because the straight line FAE meets the parallels AD, FC, the exterior angle DAE is equal to the interior and opposite angle CFA: but the angle ACF has been proved to be equal to the angle DAE; therefore also, the angle ACF is equal to the angle CFA, and, consequently, the side AF is equal to the side AC (*d*); and because AD is parallel to FC, a side of the triangle BCF, BD is to DC, as BA is to AF (*e*); but AF is equal to AC; therefore, *as* BD *is to* DC, *so is* BA *to* AC.

[2.] Next let BD be to DC, as BA is to AC, *and join* AD; then the angle CAD is equal to the angle DAE.

The same construction being made, because BD is to DC, as BA is to AC; and also BD to DC, as BA is to AF (*e*); therefore BA is to AC, as BA is to AF (*f*); wherefore AC is equal to AF (*g*); and the angle AFC equal to the angle ACF (*h*): But the angle AFC is equal to the exterior angle EAD, and the angle ACF to the alternate angle CAD; therefore *also*, EAD *is equal to the angle* CAD.

SCHOLIA. 1. This proposition consists of two theorems, the converse of each other, and is really only a second case of the third proposition. It was inserted by Dr. Simson, who imagines it to have been omitted from the Elements by some unskilful editor.

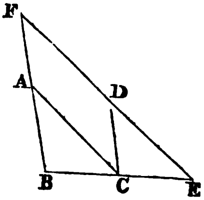
2. When the triangle ABC is isosceles, the line from the vertex which bisects the exterior angle is parallel to the base.



COROLLARY. *If both the exterior angle* (CAE) *and the adjacent interior angle* (BAC) *of a triangle be bisected by straight lines* (AD *and* AG) *which cut the base and its production*, the base thus produced is harmonically divided, that is, BD, DG, and DC are in harmonical proportion.

PROPOSITION IV.

THEOREM.—*If triangles* (ABC, DCE) *are equiangular,* [1] the sides about the equal angles are proportional; [2] and the sides which are opposite to the equal angles are homologous, that is, are the antecedents or consequents of the ratios.



DEMONSTRATION [1.] Let the triangles DCE and ABC be so placed that the sides CE and BC which are opposite to the equal angles CDE and BAC, may be contiguous and in the same straight line; then, because the angles ABC, ACB are together less than two right angles (*a*), ABC and DEC, which is equal to ACB, are also less than two right angles; wherefore BA, ED produced shall meet (*b*): *Let them be produced* and meet in the point F; and because the angle ABC is equal to the angle DCE (*c*), BF is parallel to CD (*d*). Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE (*d*); therefore FACD is a parallelogram; and consequently AF is equal to CD, and AC to FD (*e*): And because AC is parallel to FE, one of the sides of the triangle FBE, as BA is to AF, so is BC to CE (*f*); but AF is equal to CD; therefore, BA is to CD as BC is to CE (*g*); and alternately, BA *is to* BC, *as* DC *is to* CE (*h*).

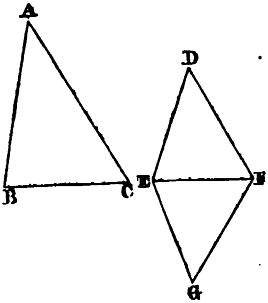
(*a*) I. 17.
(*b*) Theor. attached to I. 29.
(*c*) Hypoth.
(*d*) I. 28.
(*e*) I. 34.
(*f*) VI. 2.
(*g*) V. 7.
(*h*) V. 16.

[2.] Again, because CD is parallel to BF, BC is to CE, as FD is to DE (*f*); but FD is equal to AC; therefore BC is to CE, as AC is to DE; and alternately, BC is to CA, as CE is to ED. Therefore, because it has been proved that AB is to BC, as DC is to CE; and BC is to CA, as CE is to ED, *ex æquali*, BA *is to* AC, *as* CD *is to* DE.

SCHOLIUM. This proposition may be considered as a generalisation of the twenty-sixth proposition of the first book, the former relating to *similar* triangles, and the latter to those which are *equal*.

PROPOSITION V.

THEOREM.—*If two triangles* (ABC, DEF) *have their sides proportional* (AB to BC as DE is to EF, and BC to CA as EF is to DF), they are equiangular, and the equal angles are subtended by the homologous sides.



CONSTRUCTION. *At the points* E, F, *in the straight line* EF, *make the angle* FEG *equal to the angle* ABC (*a*), *and the angle* EFG *equal to* BCA (*a*).

DEMONSTRATION. Then the remaining angle BAC is equal to the remaining angle EGF (*b*), and the triangle ABC is therefore equiangular to the triangle GEF; and, consequently, they have their sides opposite to the equal angles proportionals (*c*). Wherefore AB is to BC, as GE is to EF; but, by the hypothesis, AB is to BC, as DE is to EF, therefore, DE is to EF, as GE is to EF. Therefore, DE and GE have the same ratio to EF (*d*), and, consequently, are equal (*e*). For the same reason, DF is equal to FG. And because, in the triangles DEF, GEF, DE is equal to EG, and EF common, and also the base DF equal to the base GF; therefore *the angle* DEF *is equal to the angle* GEF (*f*), *and the other angles to the other angles, which are subtended by the equal sides* (*g*). Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF; and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: For the same reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore *the triangle* ABC *is equiangular to the triangle* DEF.

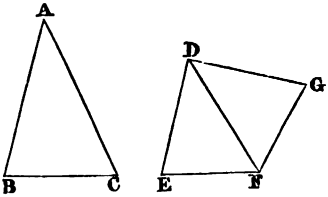
(*a*) I. 23.
(*b*) I. 32.
(*c*) VI. 4.
(*d*) V. 11.
(*e*) V. 9.
(*f*) I. 8.
(*g*) I. 4.

SCHOLIUM. This proposition is the converse of the preceding, and bears the same relation to the eighth proposition of the first book that the preceding does to the twenty-sixth of the same book.

PROPOSITION VI.

Theorem.—*If two triangles* (ABC, DEF) *have one angle in each equal* (BAC equal to EDF) *and the sides about the equal angles proportional* (BA to AC, as ED is to DF), the triangles are equiangular, and have those angles equal which the equal sides subtend.

Construction. *At the points* D, F, *in the straight line* DF, *make the angle* FDG *equal to either of the angles* BAC, EDF (*a*) ; *and the angle* DFG *equal to the angle* ACB (*a*).



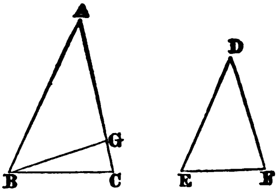
Demonstration. Then the remaining angle at B is equal to the remaining one at G (*b*), and, consequently, the triangle ABC is equiangular to the triangle DGF, and therefore BA is to AC, as GD is to DF (*c*). But, by the hypothesis, BA is to AC, as ED is to DF ; and therefore ED is to DF, as GD is to DF (*d*) ; wherefore ED is equal to DG (*e*): And DF is common to the two triangles EDF, GDF ; therefore the two sides ED, DF, are equal to the two sides GD, DF ; but the angle EDF is also equal to the angle GDF ; wherefore the base EF is equal to the base FG (*f*), and the triangle EDF to the triangle GDF, *and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides:* Therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E : But the angle DFG is equal to the angle ACB ; therefore the angle ACB is equal to the angle DFE, and the angle BAC is equal to the angle EDF (*g*) ; wherefore also the remaining angle at B is equal to the remaining angle at E. Therefore *the triangle* ABC *is equiangular to the triangle* DEF.

(*a*) I. 23.
(*b*) I. 32.
(*c*) VI. 4.
(*d*) V. 11.
(*e*) V. 9.
(*f*) I. 4.
(*g*) Hypoth.

Scholium. This proposition corresponds with the fourth proposition of the first book.

PROPOSITION VII.

THEOREM.—*If two triangles* (ABC, DEF) ***have two sides in the one proportional to two sides in the other*** (AB to BC, as DE is to EF); ***the angles*** (A and D) ***opposite to one pair of the homologous sides*** (BC and EF) ***equal; and those*** (C and F) ***opposite to the other pair, either both less, or both not less than a right angle***; the triangles are equiangular, and the angles contained by the proportional sides are equal.



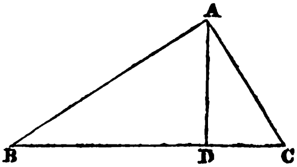
(a) I. 23.
(b) Hypoth.
(c) I. 32.
(d) VI. 4.
(e) V. 11.
(f) V. 9.
(g) I. 6.
(h) I. 17.

DEMONSTRATION. For, if the angles ABC, DEF be not equal, one of them is greater than the other; let ABC be the greater, *and at the point* B, *in the straight line* AB, *make the angle* ABG *equal to the angle* E (*a*). Because the angle at A is equal to the angle at D (*b*), and the angle ABG to the angle at E, the remaining angle AGB is equal to the remaining angle at F (*c*); therefore the triangles ABG and DEF are equiangular; wherefore as AB is to BG, so is DE to EF (*d*): But DE is to EF, as AB is to BC (*b*); therefore as AB is to BC, so is AB to BG (*e*); and because AB has the same ratio to each of the lines BC, BG; BC is equal to BG (*f*), and therefore the angle BGC is equal to the angle at C (*g*), and each of them is less than a right angle (*h*). Then since BGC is less than a right angle, BGA must be greater than a right angle, and also the angle at F which is equal to BGC; but since the angle at C is less than a right angle, the angle at F must be also less than a right angle, which is absurd; therefore the angle ABC is not greater than the angle at E, and in the same manner it may be shown that the angle at E is not greater than the angle ABC; therefore they are equal, and the angle at A is equal to the angle at D, wherefore the remaining angle at C is equal to the remaining angle at F; *and the triangles* ABC *and* DEF *are equiangular, and therefore have the sides about the equal angles proportional* (*d*).

SCHOLIUM. The demonstration of this proposition is considerably altered from Euclid, who makes three cases of it, and is unnecessarily prolix.

PROPOSITION VIII.

Theorem. In a right-angled triangle (ABC), *if a perpendicular* (AD) *be drawn from the right angle to the base;* the triangles on each side of it are similar to the whole triangle, and to one another.



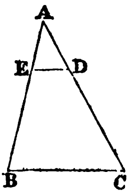
(a) I. 32.
(b) VI. 4.
(c) VI. Def. 1.

Demonstration. Because the angles BAC and ADB are equal, being both right angles, and that the angle at B is common to the two triangles ABC and ABD; the remaining angle at C is equal to the remaining angle BAD (*a*): therefore the triangles ABC and ABD are equiangular, and the sides about their equal angles are proportionals (*b*); wherefore the triangles are similar (*c*): in the like manner it may be demonstrated, that the triangles ADC and ABC are equiangular and similar: and *the triangles* ABD *and* ADC, *being both equiangular and similar to* ABC, *are equiangular and similar to each other.*

Corollary. From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base: and also that each of the sides is a mean proportional between the base, and its segment adjacent to that side: because in the triangles BDA, ADC, BD is to DA, as DA is to DC (*b*); and in the triangles ABC, DBA, BC is to BA, as BA is to BD (*b*); and in the triangles ABC, ACD, BC is to CA, as CA is to CD (*b*).

PROPOSITION IX.

Problem. From a given *finite straight line* (AB) to cut off any required part, or submultiple.



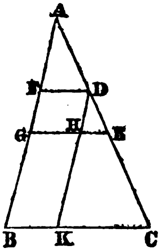
(a) VI. 2.
(b) V. 18.
(c) V. D.

Solution. *From the point* A *draw a straight line* AC, *making any angle with* AB; *and in* AC *take any point* D, *and take* AC *the same multiple of* AD, *that* AB *is of the part which is to be cut off from it; join* BC, *and draw* DE *parallel to it: then* AE *is the part required to be cut off.*

Demonstration. Because ED is parallel to BC, one of the sides of the triangle ABC; as CD is to DA, so is BE to EA (*a*); and by composition, CA is to AD, as BA is to AE (*b*): but CA is a multiple of AD; therefore BA is the same multiple of AE (*c*): whatever part, therefore, AD is of AC, AE is the same part of AB: wherefore, *from the straight line* AB *the part required is cut off.*

PROPOSITION X.

Problem. To divide a given *straight line* (AB) similarly to a given *divided straight line* (AC); that is, into parts proportional to the parts of the given divided straight line.



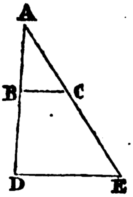
(*a*) I. 31.
(*b*) I. 34.
(*c*) VI. 2.

Solution. *Let* AC *be divided in the points* D, E; *and let* AB, AC *be placed so as to contain any angle, and join* BC, *and through the points* D, E, *draw* DF, EG, *parallels to it* (*a*); *and through* D *draw* DHK *parallel to* AB.

Demonstration. Because each of the figures FH, HB, is a parallelogram, DH is equal to FG (*b*), and HK to GB (*b*); and because HE is parallel to KC, one of the sides of the triangle DKC, as CE is to ED, so is KH to HD (*c*): but KH is equal to BG, and HD to GF; therefore, as CE is to ED, so is BG to GF: again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED is to DA, so is GF to FA: but it has been proved that CE is to ED, as BG is to GF; and as ED is to DA, so is GF to FA; therefore *the given straight line* AB *is divided similarly to* AC.

PROPOSITION XI.

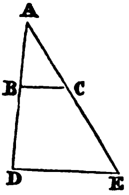
Problem. To find a third proportional to *two* given *straight lines* (AB and AC).



(a) I. 31.

Solution. *Let the two given straight lines* AB *and* AC *be so placed as to contain any angles, and produce them to the points* D, E; *make* BD *equal to* AC; *and having joined* BC, *through* D *draw* DE *parallel to it* (*a*).

Demonstration. Because BC is parallel to DE, a side of the triangle ADE, AB is to BD, as AC is to CE (*b*): but BD is equal to AC; therefore, as AB is to AC, so is AC to CE. Wherefore *to the two given straight lines* AB, AC, *a third proportional* CE *is found.*

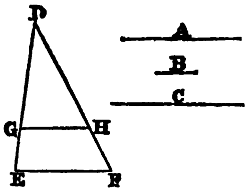


(*b*) VI. 2.

PROPOSITION XII.

Problem. To find a fourth proportional to *three* given *straight lines* (A, B, and C).

Solution. *Take two straight lines* DE, DF, *containing any angle* D, *and upon these make* DG *equal to* A, GE *equal to* B, *and* DH *equal to* C; *and having joined* GH, *draw* EF *parallel to it through the point* E (*a*); *then* HF *is the fourth proportional required.*



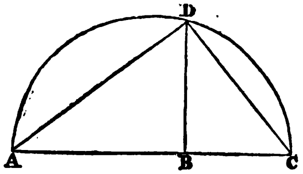
(*a*) I. 31.
(*b*) VI. 2.

Demonstration. Because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH is to HF (*b*); but DG is equal to A, GE to B, and DH to C; therefore as A is to B, so is C to HF. Wherefore, *to the three given lines,* A, B, C, *a fourth proportional* HF *is found.*

PROPOSITION XIII.

Problem. To find a mean proportional between *two* given *straight lines* (AB and BC).

Solution. *Place* AB, BC *in a straight line, and upon* AC *describe the semicircle* ADC; *from the point* B *draw* BD *at right angles to* AC (*a*), *and join* AD, DC. *Then* DB *is the mean proportional required.*



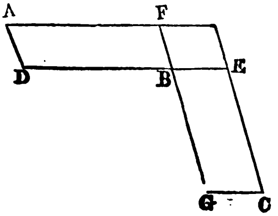
(*a*) I. 11.
(*b*) III. 31.
(*c*) VI. 8, cor.

Demonstration. Because the angle ADC in a semicircle is a right angle (*b*), and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB *is a mean proportional between* AB BC the segments of the base (*c*).

PROPOSITION XIV.

THEOREM [1.]—*If equal parallelograms* (AB and BC) *have an angle of the one equal to an angle of the other*, their sides about the equal angles are reciprocally proportional (DB is to BE, as GB is to BF).

[2.] *And if parallelograms* (AB and BC) *have an angle of the one equal to an angle of the other, and their sides about the equal angles reciprocally proportional*, they are equal to one another.



CONSTRUCTION. *Let the sides* DB, BE, *be placed contiguous, in the same straight line, with the parallelograms on opposite sides of* DE; *then* FB, BG *are in one straight line* (*a*). *Complete the parallelogram* FE.

(*a*) I. 14.
(*b*) V. 7.
(*c*) VI. 1.
(*d*) V. 11.
(*e*) V. 9.

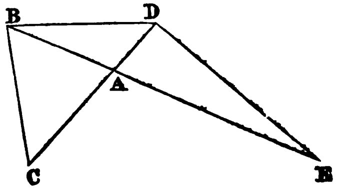
DEMONSTRATION [1.] Because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC is to FE (*b*): but as AB is to FE, so is the base DB to BE (*c*); and, as BC is to FE, so is the base GB to BF; therefore, as DB is to BE, so is GB to BF (*d*). *Wherefore the sides of the parallelograms* AB, BC, *about their equal angles are reciprocally proportional.*

[2.] Let the same construction remain, then, because DB is to BE, as GB is to BF; and DB is to BE, as the parallelogram AB is to the parallelogram FE, and GB is to BF, as the parallelogram BC is to the parallelogram FE; therefore AB is to FE, as BC is to FE (*d*): wherefore, *the parallelogram* AB *is equal to the parallelogram* BC (*e*).

PROPOSITION XV.

THEOREM [1.]—*If equal triangles* (ABC and ADE) *have an angle of the one equal to an angle of the other,* their sides about the equal angles are reciprocally proportional (CA is to AD, as EA is to AB).

[2.] *And if triangles* (ABC and ADE) *have an angle in the one equal to an angle in the other, and their sides about the equal angles reciprocally proportional,* they are equal to one another.



CONSTRUCTION. *Let the sides* CA, AD, *be placed contiguous, in the same straight line, with the triangles on opposite sides of* CD; *then* EA, AB *are in one straight line* (*a*). *Join* BD.

(*a*) I. 14.
(*b*) V. 7.
(*c*) VI. 1.
(*d*) V. 11.
(*e*) V. 9.

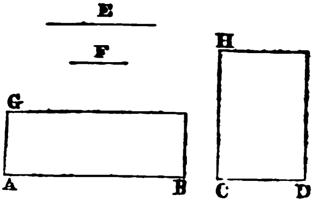
DEMONSTRATION [1.] Because the triangle ABC is equal to ADE, and that ABD is another triangle, the triangle CAB is to the triangle BAD, as the triangle EAD is to the triangle DAB (*b*): but as the triangle CAB is to the triangle BAD, so is the base CA to AD (*c*); and, as the triangle EAD is to the triangle DAB, so is the base EA to AB (*c*); therefore, as CA is to AD, so is EA to AB (*d*). Wherefore, *the sides of the triangles* ABC, ADE, *about the equal angles are reciprocally proportional.*

[2.] Let the same construction remain, then, because CA is to AD, as EA is to AB; and CA is to AD, as the triangle ABC is to the triangle BAD (*c*); and EA is to AB, as the triangle EAD is to the triangle BAD (*c*); therefore the triangle BAC is to the triangle BAD, as the triangle EAD is to the triangle BAD (*d*); wherefore *the triangle* ABC *is equal to the triangle* ADE (*e*).

PROPOSITION XVI.

THEOREM [1.]—*If four straight lines* (A B, C D, E & F) *be proportionals*, the rectangle under the extremes (A B and F) is equal in area to the rectangle under the means (C D and E).

[2.]—*And if the rectangle under the extremes be equal in area to the rectangle under the means*, the four straight lines are proportional.



(a) I. 11.
(b) V. 7.
(c) VI. 14.

CONSTRUCTION. *From the points* A, C, *draw* AG, CH, *at right angles to* AB, CD; *and make* AG *equal to* F, *and* CH *equal to* E, *and complete the parallelograms* BG, DH.

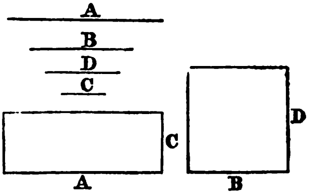
DEMONSTRATION [1.] Because AB is to CD. as E is to F; and that E is equal to CH, and F to AG; AB is to CD, as CH is to AG (*b*); therefore the sides of the parallelograms BG, DH, about the equal angles are reciprocally proportional; and therefore the parallelograms BG and DH are equal in area (*c*): and the parallelogram BG is contained by the straight lines AB, F; because AG is equal to F; and the parallelogram DH is contained by CD and E; because CH is equal to E. Therefore *the rectangle under the straight lines* AB, F, *is equal in area to that under* CD *and* E.

[2.] The same construction being made, because the rectangle under the straight lines AB, F, is equal in area to that under CD, E, and that the rectangle BG is under AB, F, because AG is equal to F; and the rectangle DH under CD, E, because CH is equal to E; therefore the parallelogram BG is equal in area to the parallelogram DH; and they are equiangular. But the sides about the equal angles of equal parallelograms are reciprocally proportional (*c*). Wherefore, AB is to CD, as CH is to AG; and CH is equal to E, and AG to F; therefore, *as* AB *is to* CD, *so is* E *to* F.

PROPOSITION XVII.

Theorem [1.]—*If three straight* lines (A, B, and C) *be proportionals,* the rectangle under the extremes (A and C) is equal in area to the square on the mean (B).

[2.] *And if the rectangle under the extremes be equal in area to the square on the mean,* the three straight lines are proportionals.



(*a*) V. 7.
(*b*) VI. 16.

Demonstration. *Take* D *equal to* B; and because A is to B, as B is to C, and that B is equal to D; A is to B, as D is to C (*a*). But if four straight lines be proportionals, the rectangle under the extremes is equal in area to that under the means (*b*). Therefore the rectangle under A, C is equal in area to that under B, D. But the rectangle under B, D is the square on B; because B is equal to D. Therefore *the rectangle under* A, C *is equal in area to the square on* B.

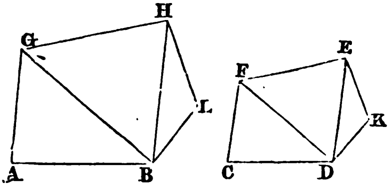
[2.] The same construction being made, because the rectangle under A, C is equal in area to the square on B, and the square on B is equal to the rectangle under B, D, because B is equal to D; therefore the rectangle under A, C is equal in area to that under B, D: But if the rectangle under the extremes be equal in area to that under the means, the four straight lines are proportionals (*b*): Therefore A is to B, as D is to C; but B is equal to D: wherefore *as* A *is to* B, *so is* B *to* C.

Scholium. The foregoing proposition is really only a particular case of the sixteenth proposition.

PROPOSITION XVIII.

Problem.—On a given *straight line* (AB) to construct a rectilineal figure similar, and similarly situated to a given *rectilineal figure.*

1. Let the given rectilineal figure be the quadrilateral (CDEF).



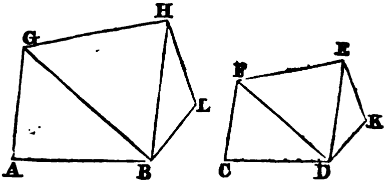
(*a*) I. 23.
(*b*) I. 32.
(*c*) VI. 4.
(*d*) V. 22.
(*e*) VI. def. 1.

Solution. *Join* DF, *and at the points* A, B, *in the straight line* AB, *form the angle* BAG *equal to the angle at* C (*a*), *and the angle* ABG *equal to the angle* CDE (*a*); *again, at the points* G, B, *in the straight line* GB, *form the angle* BGH *equal to the angle* DFE (*a*); *and the angle* GBH *equal to* FDE (*a*); *then the quadrilateral* ABHG *is similar and similarly situated to the quadrilateral* CDEF.

Demonstration. Because the angle A is equal to the angle C, and the angle ABG to CDF, therefore the remaining angle AGB is equal to the remaining angle CFD (*b*), wherefore the triangles AGB and CFD are equiangular; again, because the angle BGH is equal to the angle DFE, and the angle GBH to FDE, therefore the remaining angle GHB is equal to the remaining angle FED (*b*); wherefore the triangles BGH and DFE are equiangular. Then because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE: For the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED. Therefore the rectilineal figure ABHG is equiangular to CDEF. But likewise these figures have their sides about the equal angles proportionals; because the triangles GAB, FCD, being equiangular, BA is to AG, as DC is to CF (*c*); and because AG is to GB, as CF is to FD; and as GB to GH, so, by reason of the equiangular triangles BGH, DFE, is FD to FE; therefore, *ex æquali*, AG is to GH, as CF is to FE (*d*). In the same manner it may be proved that AB is to BH, as CD is to DE; and GH is to HB, as FE is to ED (*c*). Wherefore, because *the rectilineal figures* ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they *are similar to one another* (*e*).

2. Next let the given rectilineal figure be CDKEF.

Solution. *Join* DE, *and upon the given straight line* AB *describe the rectilineal figure* ABHG, *similar, and similarly situated to the quadrilateral figure* CDEF, *by the former case; and at the points* B, H, *in the straight line* BH, *make the angle* HBL *equal to the angle* EDK, *and the angle* BHL *equal to the angle* DEK (*a*); *then the rectilineal figure* ABLHG *is similar, and similarly situated to the figure* CDKEF.



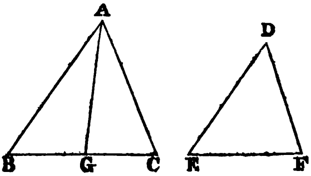
(b) I. 32.
(c) VI. 4.
(d) V. 22.
(e) VI. Def. 1.

DEMONSTRATION. Because the angle HBL is equal to EDK, and the angle BHL to EDK, therefore the remaining angle L is equal to the remaining angle K (*b*): and because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED (*e*); and the angle BHL is equal to DEK; therefore the whole angle GHL is equal to the whole angle FEK; and for the same reason the angle ABL is equal to the angle CDK: therefore the five-sided figures ABLHG, CDKEF are equiangular. And because the figures ABHG, CDEF are similar, GH is to HB, as FE is to ED (*e*); but as HB is to HL, so is ED to EK (*c*); therefore, *ex æquali*, GH is to HL, as FE is to EK (*d*); for the same reason, AB is to BL, as CD is to DK; and because the triangles BLH and DKE are equiangular, BL is to LH, as DK is to KE (*c*). Therefore, because the *rectilineal figures* ABLHG, CDKEF are equiangular, and have their sides about the equal angles proportionals, they *are similar to one another* (*e*).

SCHOLIUM. Similar figures are said to be "similarly situated" when their homologous sides are parallel.

PROPOSITION XIX.

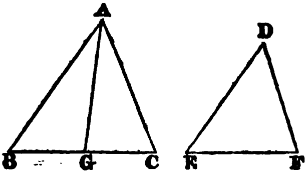
THEOREM.—*If triangles* (ABC, DEF) *are similar*, they are to one another in the duplicate ratio of their homologous sides (BC, EF).



(a) VI. 11.
(b) V. 16.
(c) V. 11.

CONSTRUCTION. *Take* BG *a third proportional to* BC, EF (*a*), *so that* BC *is to* EF, *as* EF *is to* BG: *and join* GA.

DEMONSTRATION. Then, because AB is to BC, as DE is to EF; alternately, AB is to DE, as BC is to EF (*b*); but as BC is to EF, so is EF to BG; therefore as AB is to DE, so is EF to BG (*c*); and the sides of the triangles ABG, DEF



(d) VI. 15.
(e) V. Def. 10.
(f) VI. 1.

which are about the equal angles, are reciprocally proportional. But triangles which have the sides about two equal angles reciprocally proportional are equal to one another (d); therefore the triangle ABG is equal to the triangle DEF. And because as BC is to EF, so is EF to BG; and that if three straight lines be proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second (e); therefore BC has to BG the duplicate ratio of that which BC has to EF. But as BC is to BG, so is the triangle ABC to the triangle ABG (f); therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF; but the triangle ABG is equal to the triangle DEF; wherefore also *the triangle* ABC *has to the triangle* DEF *the duplicate ratio of that which* BC *has to* EF.

COROLLARY. From this it is manifest, that *if three straight lines be proportionals,* as the first is to the third, so is any triangle upon the first to a similar, and similarly-described triangle upon the second.

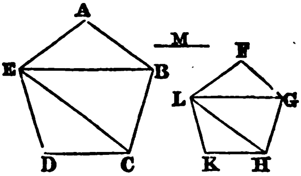
PROPOSITION XX.

THEOREM.—*If polygons* (ABCDE, FGHKL) *are similar,* they may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides (AB, FG) have.

CONSTRUCTION. *Join* BE, EC, GL, LH.

DEMONSTRATION. Because the polygon ABCDE is similar to the polygon FGHKL, the angle A is equal to the angle

F, and BA is to AE, as GF is to FL (*a*); and because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangles are equiangular (*b*), and therefore similar (*c*); wherefore the angle ABE is equal to the angle FGL. And, because the polygons are similar, the whole angle ABC is equal to the whole angle FGH (*a*); therefore the remaining angle EBC is equal to the remaining angle LGH. And because the triangles ABE, FGL are similar, EB is to BA, as LG is to GF (*a*); and also, because the polygons are similar, AB is to BC, as FG is to GH (*a*); therefore, *ex æquali*, EB is to BC, as LG is to GH (*d*); that is, the sides about the equal angles EBC, LGH are proportionals; therefore the triangles EBC and LGH are equiangular (*b*), and similar (*c*). For the same reason, the triangle ECD is similar to the triangle LHK: therefore *the similar polygons* ABCDE, FGHKL *are divided into the same number of similar triangles.*



(*a*) VI. Def. 1.
(*b*) VI 6.
(*c*) VI. 4.
(*d*) V. 22.
(*e*) VI. 19.
(*f*) V. 11.
(*g*) V. 12.

Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangles ABE, FGL, are similar, ABE has to FGL the duplicate ratio of that which the side BE has to the side GL (*e*). For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL. Therefore, as the triangle ABE is to the triangle FGL, so is the triangle BEC to the triangle GHL (*f*). Again, because the triangles EBC, LGH are similar, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH. For the same reason, the triangle ECD has to the triangle LHK the duplicate ratio of that which EC has to LH. As therefore the triangle EBC is to the triangle LGH, so is the triangle ECD to the triangle LHK (*f*); but it has been proved that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, so is the triangle EBC to the triangle LGH, and the triangle ECD to the triangle LHK. And because, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents (*g*), therefore as the triangle ABE is to the triangle

FGL, so is the polygon ABCDE to the polygon FGHKL; but the triangle ABE has to the triangle FGL the duplicate ratio of that which the side AB has to the homologous side FG. Therefore, also, *the polygon* ABCDE *has to the polygon* FGHKL *the duplicate ratio of that which* AB *has to the homologous side* FG.

Corollary 1. In like manner it may be proved, that similar four-sided figures, or figures of any number of sides, are to one another in the duplicate ratio of their homologous sides, as has already been proved in the case of triangles. Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Corollary 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has to M the duplicate ratio of that which AB has to FG (*a*); but the four-sided figure, or polygon upon AB has to the four-sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG; therefore, as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in the case of triangles (*b*). Therefore, universally, it is manifest that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly-described rectilineal figure upon the second.

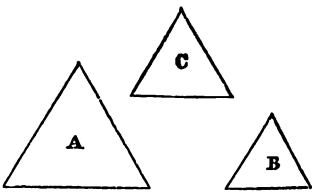
(*a*) V. Def. 10.
(*b*) VI. 19, cor.

Corollary 3. From the foregoing it follows, that the perimeters of similar rectilineal figures are as their homologous sides.

PROPOSITION XXI.

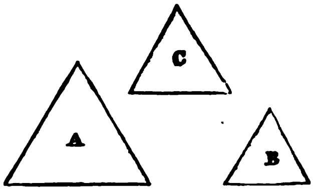
Theorem.—*If rectilineal figures* (A and B) *are similar to the same rectilineal figure* (C), they are also similar to one another.

Demonstration. Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals (*a*). Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals (*a*). Therefore the



(*a*) VI. Def. 1.

figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular (*b*), and have their sides about the equal angles proportionals (*c*). Therefore *A is similar to* B.

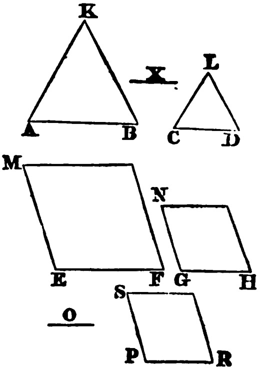


(*b*) I. Ax. 1.
(*c*) V. 11.

PROPOSITION XXII.

Theorem [1.]—*If four straight lines* (AB, CD, EF, GH) *be proportionals*, the similar rectilineal figures similarly described upon them shall also be proportionals; [2] *and if the similar rectilineal figures similarly described upon four straight lines be proportionals*, those straight lines shall also be proportionals.

Construction. *Upon* AB, CD *let the similar rectilineal figures* KAB, LCD *be similarly described; and upon* EF, GH, *the similar rectilineal figures* MF, NH, *similarly described.*



Demonstration [1.] To AB, CD, take a third proportional X (*a*); and to EF, GH, a third proportional O. And because AB is to CD, as EF is to GH, and that CD is to X, as GH is to O (*b*); therefore, *ex æquali*, as AB is to X, so is EF to O (*c*). But as AB is to X, so is the rectilineal figure KAB to the figure LCD; and as EF is to O, so is the figure MF to the figure NH (*d*); therefore, *as* KAB *is to* LCD, *so is* MF *to* NH (*b*).

[2.] *Take a line* PR, *so that* AB *is to* CD, *as* EF *is to* PR (*e*), *and upon* PR *describe the rectilineal figure* SR *similar and similarly*

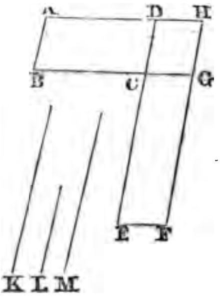
(*a*) VI. 11.
(*b*) V. 11.
(*c*) V. 22.
(*d*) VI. 20, cor. 2.
(*e*) VI. 12.
(*f*) VI. 18.
(*g*) V. 9.
(*h*) V. 7.

situated to either of the figures MF, NH (*f*). Then because as AB is to CD, so is EF to PR, and that upon AB, CD are described the similar and similarly-situated rectilineal figures KAB, LCD, and upon EF, PR, in like manner the similar rectilineal figures MF, SR; KAB is to LCD, as MF is to SR; and therefore the rectilineal figure MF having the same ratio to each of the two NH, SR, these are equal to one another (*g*); they are also similar and similarly situated; therefore GH is equal to PR. And because as AB is to CD, so is EF to PR, and that PR is equal to GH (*h*); AB *is to* CD, *as* EF *is to* GH.

PROPOSITION XXIII.

Theorem.—*If parallelograms* (AC, CF) *are equiangular,* they have to one another the ratio which is compounded of the ratios of their sides.

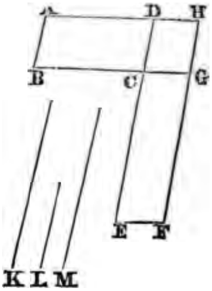
Construction. *Let* BC, CG, *two of the sides about the equal angles be placed in a straight line; therefore* DC *and* CE *are also in a straight line* (*a*). *Complete the parallelogram* DG; *and, taking any straight line* K, *make as* BC *is to* CG, *so is* K *to* L; *and as* DC *is to* CE, *so make* L *to* M (*b*).



Demonstration. The ratios of K to L, and L to M, are the same with the ratios of the sides, namely, of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to M (*c*); wherefore also K has to M the ratio compounded of the ratios of the sides. And because as BC is to CG, so is the parallelogram AC to the parallelogram CH (*d*); and as BC is to CG, so is K to L; therefore K is to L, as the parallelogram AC is to the parallelogram CH (*e*). Again, because as DC is to CE, so is the parallelogram CH to the parallelogram CF; and as DC is to CE, so is L to M; therefore L is to M, as the paral-

(*a*) I. 14.
(*b*) VI. 12.
(*c*) V. Def. 12.
(*d*) VI. 1.
(*e*) V 11.

lelogram CH is to the parallelogram CF. Therefore, since it has been proved, that as K is to L, so is the parallelogram AC to the parallelogram CH; and as L is to M, so is the parallelogram CH to the parallelogram CF; *ex æquali*, K is to M, as the parallelogram AC is to the parallelogram CF (*f*). But K has to M the ratio which is compounded of the ratios of the sides; therefore also *the parallelogram* AC *has to the parallelogram* CF *the ratio which is compounded of the ratios of the sides.*



(*f*) V. 22.

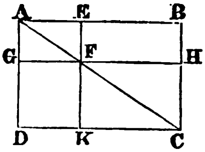
Corollary 1. *If triangles have an angle of the one equal to an angle of the other*, they are to one another as the rectangles under the sides about those angles.

Corollary 2. *If triangles and parallelograms are equiangular*, they are to one another as the rectangles under their bases and altitudes.

PROPOSITION XXIV.

Theorem.—*If parallelograms* (EG, HK) *are about the diameter of any parallelogram* (ABCD), they are similar to the whole and to one another.

Demonstration. Because DC, GF are parallels, the angle D is equal to the angle AGF (*a*); and because BC, EF are parallels, the angle B is equal to the angle AEF (*a*); also each of the angles BCD, EFG are equal to the opposite angle DAB (*b*), and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular. And because in the triangles BAC, EAF the angles B and AEF are equal, and the angle BAC common to both, they are equiangular to one another; therefore as AB is to BC, so is AE to EF (*c*). And because the opposite sides of parallelograms are equal to one another (*b*), AB is to AD, as AE is to AG (*d*); and DC is to CB, as GF is to FE; and also CD is to DA, as FG is to GA. Therefore

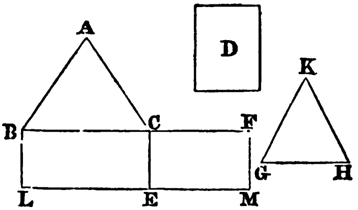


(*a*) I. 29.
(*b*) I. 34.
(*c*) VI. 4.
(*d*) V. 7.
(*e*) VI. Def. 1.
(*f*) VI. 21.

the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore similar to one another (*e*); and for the same reason, the parallelogram ABCD is similar to the parallelogram FHCK. Wherefore *each of the parallelograms* GE, KH, *is similar to* DB; but rectilineal figures which are similar to the same rectilineal figure are also similar to one another (*f*); therefore *the parallelogram* GE *is similar to* KH.

PROPOSITION XXV.

Problem. To construct a rectilineal figure which shall be similar to *one* (ABC), and equal to *another given rectilineal figure* (D).



Solution. *Upon the straight line* BC *construct the parallelogram* BE *equal to the figure* ABC (*a*); *also upon* CE *construct the parallelogram* CM *equal to* D, *and having the angle* FCE *equal to the angle* CBL (*a*); *then* BC *and* CF *are in a straight line* (*b*), *as are also* LE *and* EM. *Between* BC *and* CF *find a mean proportional* GH (*c*), *and upon* GH *construct the rectilineal figure* KGH, *similar and similarly situated to the figure* ABC (*d*), *and it shall be the rectilineal figure required equal to* D.

(*a*) I. 45, cor.
(*b*) I. 29 and 14.
(*c*) VI. 13.
(*d*) VI. 18.
(*e*) VI. 20, cor. 2.
(*f*) VI. 1.
(*g*) V. 11.
(*h*) V. 14.

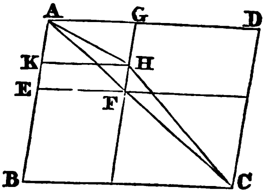
Demonstration. For BC is to GH, as GH is to CF, and if three straight lines be proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly-described figure upon the second (*e*); therefore as BC is to CF, so is the rectilineal figure ABC to KGH; but as BC is to CF, so is the parallelogram BE to the parallelogram EF (*f*); therefore as the rectilineal figure ABC is to KGH, so is the parallelogram BE to EF (*g*). But the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal to the parallelogram EF (*h*). But EF is equal to the figure D; therefore also KGH *is equal to* D; *and it is similar to* ABC.

SCHOLIUM. This proposition may be more generally enunciated "To construct a figure of a given *species* and a given *magnitude*."

PROPOSITION XXVI.

THEOREM. *If two similar parallelograms* (ABCD, AEFG) *have a common angle* (DAB), *and be similarly situated,* they are about the same diameter.

DEMONSTRATION. For, if not, let the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; *through* H *draw* HK *parallel to* AD *or* BC (*a*). Then, because the parallelograms ABCD, AKHG are about the same diameter, they are similar to one another (*b*); therefore as DA is to AB, so is GA to AK (*c*). But because ABCD and AEFG are similar parallelograms, as DA is to AB, so is GA to AE; therefore GA is to AE, as GA is to AK (*d*); wherefore GA has the same ratio to each of the straight lines AE, AK, therefore they are equal (*e*), the less to the greater, which is impossible. Therefore ABCD and AKHG are not about the same diameter; wherefore ABCD *and* AEFG *must be about the same diameter.*

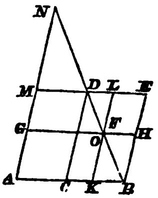


(*a*) I. 31.
(*b*) VI. 24.
(*c*) VI. Def. 1.
(*d*) V. 11.
(*e*) V. 9.

PROPOSITION XXVII.

THEOREM.—Of all the parallelograms that can be inscribed in any triangle (NAB), *that which is constructed on the half of one of the sides as base,* is the greatest.

CONSTRUCTION. *Let* ACDM *be a parallelogram constructed on half the base* AB, *and* AKFG *any other parallelogram inscribed in the triangle* NAB; *complete the parallelogram* AE *and produce* GF *and* KF *to* L *and* H.



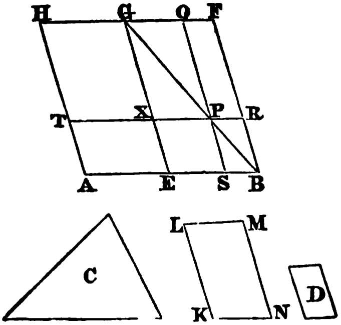
(*a*) I. 36.
(*b*) I. 43.

Demonstration. Because AC and CB are equal, the parallelogram AD is equal to the parallelogram CE, and the parallelogram AO to the parallelogram CH (*a*); and because CF and FE are the complements of the parallelograms OL and KH, CF is equal to FE (*b*); therefore, adding equals to equals, the parallelogram AF is equal to the gnomon LBC. But the parallelogram CE is greater than the gnomon LBC; therefore the parallelogram AB is also greater than the gnomon LBC. But the parallelogram AF is equal to the gnomon LBC; therefore the parallelogram AD is greater than the parallelogram AF; and in the same manner it may be shown *that the parallelogram* AD *is greater than any other parallelogram that can be inscribed in the triangle* NAB.

Scholium. The enunciation of this proposition, as given by Euclid, is as follows:—"Of all the parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest." That which has been substituted above is not only more intelligible but admits of a shorter prove

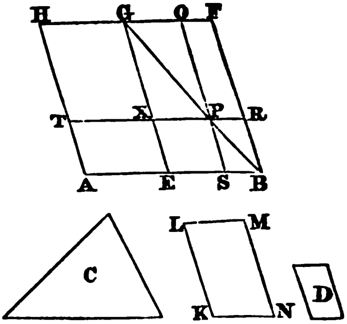
PROPOSITION XXVIII.

Problem. To divide a given *straight line* (AB) into two parts such that parallelograms of equal altitude may be constructed upon them, one equal to a given *rectilineal figure* (C), and the other similar to a given *parallelogram* (D); the rectilineal figure (C) not being greater than the parallelogram constructed on half the given line, and similar to the given parallelogram.



Solution. *Divide* AB *into two equal parts in the point* E (*a*), *and upon* EB *construct the parallelogram* EBFG *similar and similarly situated to* D (*b*), *and complete the parallelogram* AC, *which, by the determination, must be either*

(*a*) I. 10.
(*b*) VI. 18.

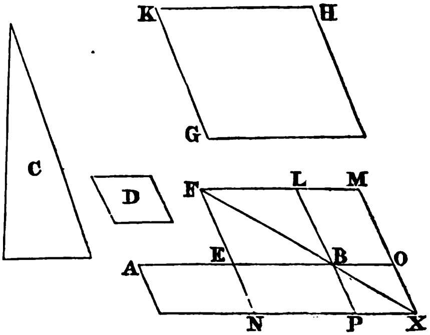


equal to C, *or greater than it. If* AG *be equal to* C, *then what was required is already done.* For, upon AE, one of the parts of AB, the parallelogram AG is constructed equal to the given rectilineal figure C; and upon EB, the other part, a parallelogram of equal altitude, has been constructed, similar and similarly situated to the given parallelogram D. But if AG be not equal to C, it is greater than it; and because EF is equal to AG, therefore EF is also greater than C. *Make the parallelogram* KLMN *equal to the excess of* EF *above* C, *and similar and similarly situated to* D (*c*); but D is similar to EF, therefore also the parallelogram KM is similar to EF (*d*). Let KL be the homologous side to EG, and LM to GF; then because EF is equal to C and KM together, EF is greater than KM, therefore the straight line EG is greater than KL, and GF than LM. *Make* GX *equal to* LK, *and* GO *equal to* LM, *and complete the parallelogram* XGOP. Then XO is equal and similar to KM; but KM is similar to EF; therefore also XO is similar to EF, and therefore XO and EF are about the same diagonal (*e*). Let GPB be their diagonal, *and produce* XP *to* T *and* R, *and* OP *to* S. Then because EF is equal to C and KM together, and XO a part of the one is equal to KM a part of the other, the remainder, namely, the gnomon ORE, is equal to the remainder C. And because OR is equal to XS (*f*), by adding SR to each, the whole OB is equal to the whole XB; but XB is equal to TE (*g*), because the bases AE and EB are equal; wherefore also TE is equal to OB; add XS to each, then the whole TS is equal to the gnomon ORE; but it has been proved that the gnomon ORE is equal to C; and therefore also TS is equal to C. Wherefore *the given line* AB *is divided into two parts* AS, SB, *such that the parallelogram* TS *constructed on one of them is equal to* C, *and the parallelogram* SR *of the same altitude, constructed on the other part, is similar to the given one* D, because SR is similar to EF (*h*).

(*c*) VI. 25.
(*d*) VI. 21.
(*e*) VI. 26.
(*f*) I. 34.
(*g*) I. 36.
(*h*) VI. 24.

PROPOSITION XXIX.

PROBLEM. To produce a given *straight line* (AB) so that a parallelogram similar to a given *one* (D) being constructed on the produced part, another parallelogram of equal altitude constructed on the whole line produced, may be equal to a given rectilineal figure (C).



SOLUTION. *Bisect* AB *in the point* E, *and on* EB *construct the parallelogram* EL, *similar and similarly situated to* D (*a*); *and make the parallelogram* GH *equal to* EL *and* C *together, and similar and similarly situated to* D (*b*); wherefore GH is similar to EL (*c*). Let KH be the side homologous to FL, and KG to FE; then because the parallelogram GH is greater than EL, therefore the side KH is greater than FL, and EG than FE. *Produce* FL *and* FE, *and make* FLM *equal to* KH, *and* FEN *to* KG, *and complete the parallelogram* MN. MN is therefore equal and similar to GH; but GH is similar to EL; wherefore MN is similar to EL, and consequently EL and MN are about the same diagonal (*d*). *Draw their diagonal* FX, *and complete the figure.* Therefore since GH is equal to EL and C together, and that GH is equal to MN; MN is equal to EL and C; take away the common part EL; then the remainder, namely, the gnomon NOL, is equal to C. And because AE is equal to EB, the parallelogram AN is equal to the parallelogram NB (*e*), that is, to BM (*f*). Add NO to each, therefore the whole parallelogram AX, is equal to the gnomon NOL. But the gnomon NOL is equal to C. Wherefore, *upon the whole produced line* AO *there is constructed the parallelogram* AX *equal to the given figure* C; *and the parallelogram* PO *of the same altitude as* AX, *is constructed on the produced part* BO, *and is similar to* D, because PO is similar to EL.

(*a*) VI. 18.
(*b*) VI. 25.
(*c*) VI. 21.
(*d*) VI. 26.
(*e*) I. 36.
(*f*) I. 43.
(*g*) VI. 24.

PROPOSITION XXX.

Problem. To cut a given *straight line* (AB) in extreme and mean ratio.

Solution. *Divide* AB *in the point* C, *so that the rectangle under* AB, BC, *may be equal to the square on* AC (*a*).

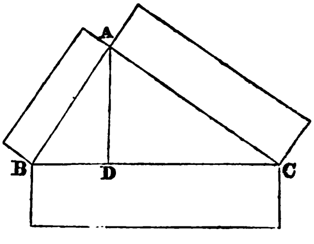
(*a*) II. 11.
(*b*) VI. 17.
(*c*) VI. Def. 3.

Demonstration. Then, because the rectangle AB, BC is equal to the square on AC, as BA is to AC, so is AC to CB (*b*); therefore AB *is cut in extreme and mean ratio in* C (*c*).

PROPOSITION XXXI.

Theorem.—If a triangle (ABC) be right-angled, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly-described figures upon the sides containing the right angle.

Demonstration. *Draw the perpendicular* AD (*a*). Then, because in the right-angled triangle ABC, AD is drawn from the right angle at A, perpendicular to the base BC, the triangles ABD, ADC are similar to the whole triangle ABC, and to one another (*b*); and because the triangle ABC is similar to ADB, as CB is to BA, so is BA to BD (*c*); and because these three straight lines are proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly-described figure upon the second (*d*); *therefore as CB is to BD*, so

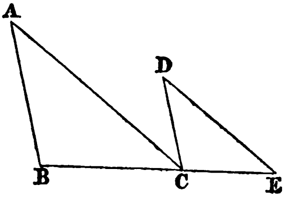


(*a*) I. 12.
(*b*) VI. 8.
(*c*) VI. 4.
(*d*) VI. 20, cor. 2.
(*e*) V. B.
(*f*) V. 24.
(*g*) V. A.

is the figure upon CB to the similar and similarly-described figure upon BA: and inversely, as DB is to BC, so is the figure upon BA to that upon BC (*e*): for the same reason, as DC is to CB, so is the figure upon CA to that upon CB: therefore as BD and DC together are to BC, so are the figures upon BA, AC to that upon BC (*f*): but BD and DC together are equal to BC; therefore *the figure described on* BC *is equal to the similar and similarly-described figures upon* BA, AC (*g*).

PROPOSITION XXXII.

THEOREM.—*If two triangles* (ABC, DCE) *which have two sides of the one* (BA, AC) *proportional to two sides of the other* (CD, DE) *be joined at one angle so as to have their homologous sides parallel to one another*, the remaining sides shall be in a straight line.



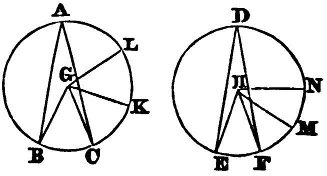
(*a*) I. 29.
(*b*) VI. 6.
(*c*) I. 32.
(*d*) I. 14.

DEMONSTRATION. Because AB is parallel to DC, and the straight line AC meets them, the alternate angles A, ACD are equal (*a*): for the same reason, the angle D is equal to the angle ACD; wherefore also A is equal to D: and because the triangles ABC, DCE have one angle at A equal to one at D, and the sides about these angles proportionals, viz. BA to AC, as CD is to DE, the triangle ABC is equiangular to the triangle DCE (*b*); therefore the angle B is equal to the angle DCE: and the angle A was proved to be equal to ACD; therefore the whole angle ACE is equal to the two angles B, A: add the common angle ACB, then the angles ACE, ACB are equal to the angles B, A, ACB: but B, A, ACB are equal to two right angles (*c*): therefore also the angles ACE, ACB are equal to two right angles; and since at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles, therefore BC and CE are in a straight *line* (*d*).

PROPOSITION XXXIII.

THEOREM.—*In equal circles, angles, whether at the centres or circumferences*, have the same ratio which the circumferences on which they stand have to one another; so also have the sectors.

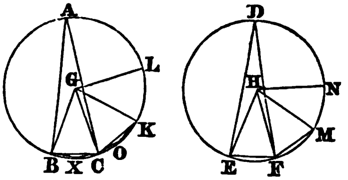
DEMONSTRATION. Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences: as the circumference BC to the circumference EF, so shall the angle BGC be to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

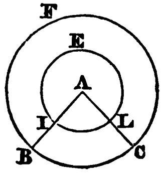


(*a*) III. 27.
(*b*) V. Def. 5.
(*c*) V. 15.
(*d*) III. 20.
(*e*) I. 4.
(*f*) III. Def. 11.
(*g*) III. 24.

Take any number of circumferences CK, KL, *each equal to* BC, *and any number whatever* FM, MN, *each equal to* EF; *and join* GK, GL, HM, HN. Because the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are also all equal (*a*); therefore what multiple soever the circumference BL is of the circumference BC, the same multiple is the angle BGL of the angle BGC: for the same reason, whatever multiple the circumference EN is of the circumference EF, the same multiple is the angle EHN of the angle EHF: and if the circumference BL be equal to the circumference EN, the angle BGL is also equal to the angle EHN (*a*); and if the circumference BL be greater than EN, likewise the angle BGL is greater than EHN; and if less, less: therefore, since there are four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; and that of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the circumference EN, and the angle EHN; and since it has been proved, that if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less: therefore as the circumference BC is to the circumference EF, so is the angle BGC to the angle EHF (*b*): but as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF (*c*); for each is double of each (*d*); therefore *as the circumference* BC *is to* EF, *so is the angle* BGC *to the angle* EHF, *and the angle* BAC *to the angle EDF.*

Also, as the circumference BC is to EF, so shall the sector BGC be to the sector EHF. *Join* BC, CK, *and in the circumferences* BC, CK *take any points* X, O, *and join* BX, XC, CO, OK: then, because in the triangles GBC, GCK, the two sides BG, GC are equal to the two CG, GK, each to each, and that they contain equal angles (*a*); the base BC is equal to the base CK, and the triangle GBC to the triangle GCK (*e*): and because the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle ABC is equal to the remaining part of the whole circumference of the same circle: therefore the angle BXC is equal to the angle COK (*a*); and the segment BXC is therefore similar to the segment COK (*f*); and they are upon equal straight lines, BC, CK: but similar segments of circles upon equal straight lines are equal to one another (*g*); therefore the segment BXC is equal to the segment COK: and the triangle BGC was proved to be equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK: for the same reason, the sector KGL is equal to each of the sectors, BGC, CGK: in the same manner, the sectors EHF, FHM, MHN may be proved equal to one another: therefore, what multiple soever the circumference BL is of the circumference BC, the same multiple is the sector BGL of the sector BGC; and for the same reason, whatever multiple the circumference EN is of EF, the same multiple is the sector EHN of the sector EHF; and if the circumference BL be equal to EN, the sector BGL is equal to the sector EHN; and if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less: since then, there are four magnitudes, the two circumferences BC, EF, and the two sectors BGC, EHF; and that of the circumference BC, and sector BGC, the circumference BL and sector BGL are any equimultiples whatever; and of the circumference EF, and sector EHF, the circumference EN, and sector EHN are any equimultiples whatever; and since it has been proved, that if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if equal, equal; and if less, less: therefore, *as the circumference* BC *is to the circumference* EF, *so is the sector BGC to the sector* EHF (*b*).

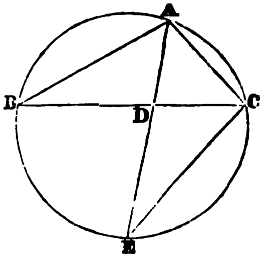




PROPOSITION B.

THEOREM.—*If an angle* (BAC) *of a triangle* (ABC) *be bisected by a straight line* (AD) *which likewise cuts the base*, the rectangle under the sides of the triangle (BA, AC) is equal to the rectangle under the segments of the base (BD, DC) together with the square on the straight line (AD) which bisects the angle.

DEMONSTRATION. *Describe the circle* ACB *about the triangle* (*a*), *and produce* AD *to the circumference in* E, *and join* EC: then because the angle BAD is equal to the angle CAE, and the angle ABD to the angle AEC, for they are in the same segment (*b*), the triangles ABD, AEC are equiangular to one another (*c*): therefore as BA is to AD, so is EA to AC (*d*); and consequently the rectangle BA, AC is equal to the rectangle EA, AD (*e*); that is, to the rectangle ED, DA, together with the square on AD, (*f*): but the rectangle ED, DA is equal to the rectangle BD, DC (*g*); therefore *the rectangle* BA, AC *is equal to the rectangle* BD, DC, *together with the square on* AD.

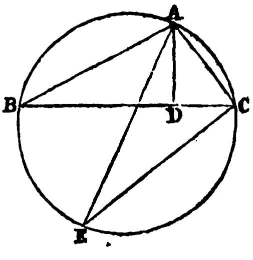


(*a*) IV. 5.
(*b*) III. 21.
(*c*) I. 32.
(*d*) VI. 4.
(*e*) VI. 16.
(*f*) II. 3.
(*g*) III. 35.

PROPOSITION C.

THEOREM.—*If from any angle* (A) *of a triangle* (ABC) *a straight line* (AD) *be drawn perpendicular to the base* (BC), the rectangle, under the sides of the triangle (BA, AC) is equal to the rectangle under the perpendicular (AD) and the diameter of the circle described about the triangle.

DEMONSTRATION. *Describe the circle* ACB *about the triangle* (*a*), *and draw its diameter* AE, *and join* EC: because the right angle BDA is equal to the angle ECA in a semicircle (*b*), and the angle ABD equal to the angle AEC in the same segment (*c*); the triangles ABD, AEC are equiangular: therefore, as BA is to AD, so is EA to AC (*d*); and consequently *the rectangle* BA, AC is equal to the rectangle EA, AD (*e*).



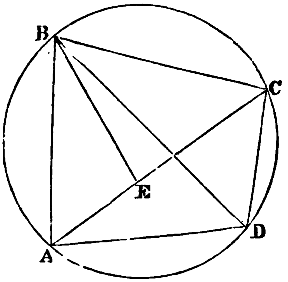
(*a*) VI. 5.
(*b*) III. 21.
(*c*) III. 31.
(*d*) VI. 4.
(*e*) VI. 16.

PROPOSITION D.

THEOREM.—*The rectangle under the diagonals of a quadrilateral figure inscribed in a circle,* is equal to both the rectangles contained by its opposite sides.

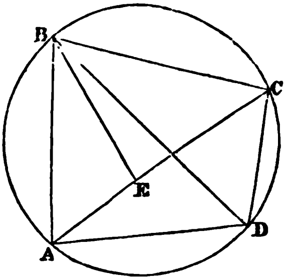
DEMONSTRATION. Let ABCD be any quadrilateral figure inscribed in a circle, *and join* AC, BD: the rectangle contained by AC, BD shall be equal to the two rectangles contained by AB, CD, and by AD, BC.

Make the angle ABE *equal to the angle* DBC (*a*); add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC: and the angle BDA is equal to the angle BCE, because they are in the same segment (*b*); therefore the triangle ABD is equiangular to the triangle BCE: wherefore, as BC is to CE, so is BD to DA (*c*); and consequently the rectangle BC, AD is equal to the rectangle BD, CE (*d*): again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC (*b*), the triangle ABE is equiangular to the triangle BCD; therefore as BA is to AE, so is BD



(*a*) I. 23.
(*b*) III. 21.
(*c*) VI. 4.
(*d*) VI. 16.

to DC (*c*); wherefore the rectangle BA, DC is equal to the rectangle BD, AE (*d*): but the rectangle BC, AD has been shown equal to the rectangle BD, CE; therefore the rectangles BC, AD, and BA, DC are together equal to the rectangles BD, CE, and BD, AE; that is, to the whole rectangle BD, AC (*e*); therefore *the whole rectangle* AC, BD *is equal to the rectangle* AB, DC, *together with the rectangle* AD, BC.



(*c*) VI. 4.
(*d*) VI. 16.
(*e*) II. 1.

Scholium. This proposition is a Lemma of Cl. Ptolomæus, in page 9 of the Μεγάλη Σύνταξις, or "*Great Construction.*"

THE

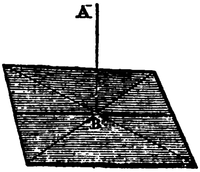
ELEMENTS OF EUCLID.

BOOK XI.

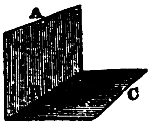
DEFINITIONS.

1. A Solid is a magnitude, *having length, breadth, and thickness.*

Corollary. All solids are bounded by *superficies*, or *surfaces*.

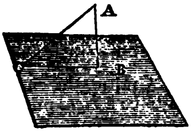


2. A straight line AB is said to be *perpendicular* to a plane, when it makes right angles with all straight lines which meet it in that place.



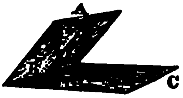
3. A plane is said to be *perpendicular* to a plane, when any straight line AB, drawn in one of the planes perpendicular to the common section of the two planes, is perpendicular to the other plane.

Scholium. The *common section* of two planes is the line in which they mutually cut or intersect each other.



4. The *inclination* of a straight line AC to a plane is the acute angle C formed by that straight line, and another CB drawn from the point C, in which the first line meets the plane, to the point B in which a perpendicular AB to the plane drawn from any point A of the first line above the plane, meets the same plane.

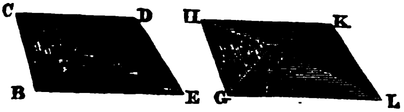
5. The *inclination* of one plane to another is the acute angle ABC, formed by two straight lines drawn from any the same point B of their common section at right angles to it, one AB upon one plane, and the other BC upon the other plane.



6. **Parallel planes** are such as do not meet one another, though produced ever so far in every direction.

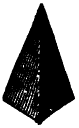
7. A **Solid angle** is that which is made by the meeting in one point of more than two plane angles, which are not in the same plane.

8. **Equal and similar solid figures**, CBED, HGLK, are such as are contained by similar planes equal in number, magnitude, and inclination to one another.



9. **Similar solid figures** are such as have all their solid angles equal, each to each, and are contained by the same number of planes similarly situated.

10. A **pyramid** is a solid figure contained by planes that are constituted between one plane figure and a point above it.



Scholium. The last-named plane figure is called the *base*, and the point above it the *vertex* of the pyramid; and all the planes meeting together in the vertex are *triangles*. The *altitude* of a pyramid is the perpendicular drawn from its vertex to its base.

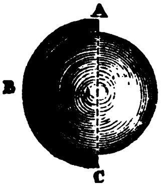
11. A **prism** is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms.

Scholia. 1. The opposite ends are termed the *bases* of the prism, and the parallelograms its *sides;* but the term *base* is sometimes applied to any side upon which it is supposed to stand. The *altitude* of a prism is a perpendicular from one of its ends or bases to the other.

2. A prism, the ends or bases of which are perpendicular to its sides, is said to be a *right* prism; any other is an *oblique* prism.

3. Pyramids and prisms are said to be *triangular*, *quadrangular*, *pentagonal*, or *polygonal*, according as their bases are triangles, quadrangles, pentagons, or polygons.

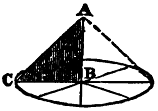
12. A SPHERE is a solid figure described by the revolution of a semicircle (ABC) about its diameter (AC), which remains unmoved.

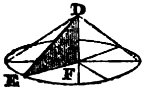


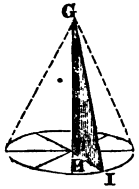
13. The AXIS OF A SPHERE is the fixed straight line (AC) about which the semicircle revolves.

14. The CENTER OF A SPHERE is the same with that of the generating semicircle.

15. The DIAMETER OF A SPHERE is any straight line which passes through its center, and is terminated both ways by the superficies of the sphere.





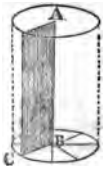


16. A CONE is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. If the fixed side (AB) be equal to the other side containing the right angle (CB), the cone is said to be *right-angled;* if it (DF) be less than the other side (EF), *obtuse-angled;* and if greater (as GH and HI) *acute-angled.*

17. The AXIS OF A CONE is the fixed straight line about which the triangle revolves.

18. The BASE OF A CONE is the circle described by that side containing the right angle, which revolves.

19. A CYLINDER is a solid figure described by the revolution of a right-angled parallelogram (ABC) about one of its sides (AB), which remains fixed.

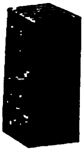


20. The AXIS OF A CYLINDER is the fixed straight line (AB) about which the parallelogram revolves.

21. The BASES OF A CYLINDER are the circles described by the two revolving opposite sides of the parallelogram.

22. **Similar cones and cylinders** are those which have their axes and the diameters of their bases proportionals.

23. A **parallelopiped** is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

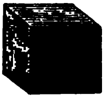


Scholium. A parallelopiped is a prism with parallelograms for its base. When its sides are rectangles it is said to be *right*, if otherwise, *oblique*.

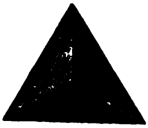
24. A **polyhedron** is a solid figure contained by plane figures.

Scholium. When all the plane figures are equal and similar, the polyhedron is said to be *regular*.

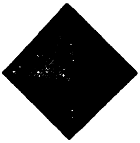
25. A **cube**, or **hexahedron**, is a solid figure contained by six equal squares.



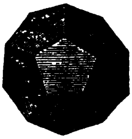
26. A **tetrahedron** is a solid figure contained by four equal and equilateral triangles.



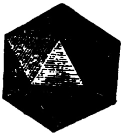
27. An **octahedron** is a solid figure contained by eight equal and equilateral triangles.



28. A **dodecahedron** is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



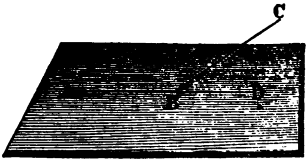
29. An **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.



PROPOSITION I.

Theorem.—One part of a straight line cannot be in a plane, *if another part is above it.*

Demonstration. If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it: and since the straight line AB is in the plane, it can be produced in that plane: *let it be produced to* D; and let any plane pass through the straight line AD, and be turned about it until it pass through the point C; and because the points B, C are in this plane, the straight line (*a*) BC is in it: therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB: which is impossible (*b*). Therefore AB *and* CD *are in the same plane.*

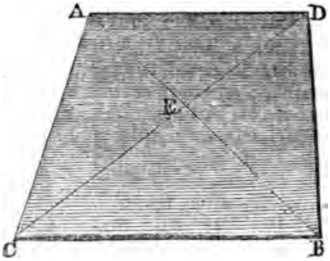


(*a*) I. Def. 6.
(*b*) I. 11 cor.

PROPOSITION II.

Theorem.—*If two straight lines* (AB, CD) *cut one another,* they are in one plane; *and if three straight lines* (EC, CB, BE) *meet one another,* they are in one plane.

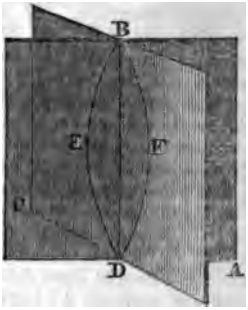
Demonstration. Let any plane pass through the straight line EB, and let the plane be turned about EB, produced if necessary, until it pass through the point C: then, because the points E, C are in this plane, the straight line EC is in it (*a*); for the same reason, the straight line BC is in the same; and by the hypothesis, EB is in it; therefore *the three straight lines* EC, CB, BE *are in one plane:* but in the plane in which EC, EB are, in the same are CD, AB (*b*); therefore AB, CD *are in one plane.*



(*a*) I. Def. 6.
(*b*) XI. 1

PROPOSITION III.

THEOREM.—*If two planes* (AB, BC) *cut one another*, their common section (DB) is a straight line.

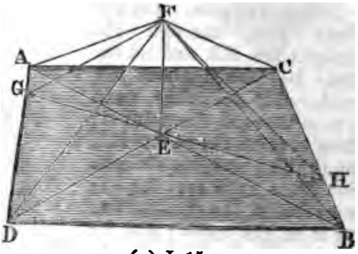


DEMONSTRATION. If it be not, *from the point* D *to* B, *draw, in the plane* AB, *the straight line* DEB, *and in the plane* BC, *the straight line* DFB: then two straight lines DEB, DFB have the same extremities, and therefore include a space betwixt them; which is impossible (*a*); therefore BD, the common section of the planes AB, BC, *cannot but be a straight line.*

(*a*) I. Ax. 10.

PROPOSITION IV.

THEOREM.—*If a straight line* (EF) *stand at right angles to each of two straight lines* (AB, CD) *in the point of their intersection* (E), it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.



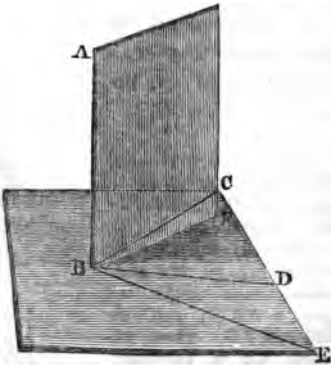
DEMONSTRATION. Take the straight lines AE, EB, CE, ED, all equal to one another; *and through* E, *draw, in the plane in which are* AB, CD, *any straight line* GEH, *and join* AD, CB: *then from any point* F, *in* EF, *draw* FA, FG, FD, FC, FH, FB: and because the two straight lines AE, ED are equal to the two BE, EC, each to each, and that they contain equal angles AED, BEC (*a*), the base AD is

(*a*) I. 15.
(*b*) I. 14.
(*c*) I. 26.
(*d*) I. 8.
(*e*) I. Def. 9.
(*f*) XI. Def. 3.

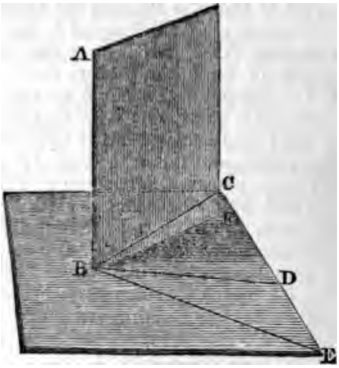
equal to the base BC, and the angle DAE to the angle EBC (*b*): and the angle AEG is equal to the angle BEH (*a*); therefore the triangles AEG, BEH have two angles of the one, equal to two angles of the other, each to each, and the sides AE, EB, adjacent to the equal angles, equal to one another; wherefore they have their other sides equal (*c*); therefore GE is equal to EH, and AG to BH: and because AE is equal to EB, and FE common and at right angles to them, the base AF is equal to the base FB (*b*); for the same reason, CF is equal to FD: and because AD is equal to BC, and AF to FB, the two sides FA, AD are equal to the two FB, BC, each to each; and the base DF was proved equal to the base FC; therefore the angle FAD is equal to the angle FBC (*d*); again, it was proved that GA is equal to BH, and also AF to FB; therefore FA and AG, are equal to FB and BH, each to each; and the angle FAG has been proved equal to the angle FBH; therefore the base GF is equal to the base FH (*b*): again, because it was proved that GE is equal to EH, and EF is common, therefore GE, EF are equal to HE, EF, each to each; and the base GF is equal to the base FH; therefore the angle GEF is equal to the angle HEF (*d*); and consequently each of these angles is a right angle (*e*); therefore FE makes right angles with GH, that is, with any straight line drawn through E, in the plane passing through AB, CD. In like manner it may be proved, that FE makes right angles with every straight line which meets it in that plane. But a straight line is at right angles to a plane when it makes right angles with every straight line which meets it in that plane (*f*): therefore EF is at right angles *to the plane in which are* AB, CD.

PROPOSITION V.

THEOREM. — *If three straight lines* (BC, BD, BE) *meet all in one point* (B), *and a straight line* (AB) *stand at right angles to each of them in that point*, these three straight lines are in one and the same plane.



DEMONSTRATION. If not, let, if it be possible, BD and BE be in one plane, and BC be above it; and let a plane pass through AB, BC, the

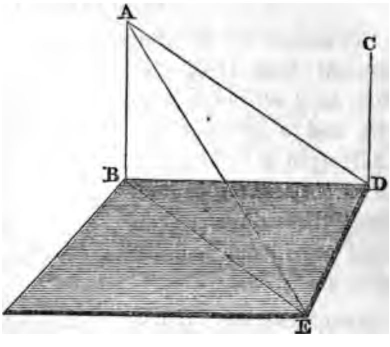


common section of which, with the plane in which BD and BE are, is a straight line (*a*); let this be BF: therefore the three straight lines AB, BC, BF are all in one plane, viz. that which passes through AB, BC: and because AB stands at right angles to each of the straight lines BD, BE, it is also at right angles to the plane passing through them (*b*); and therefore makes right angles with every straight line in that plane which meets it (*c*): but BF, which is in that plane, meets it; therefore the angle ABF is a right angle: but the angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane; which is impossible (*d*): therefore the straight line BC is not above the plane in which are BD and BE: wherefore *the three straight lines* BC, BD, BE *are in one and the same plane.*

(*a*) XI. 3.
(*b*) XI. 4.
(*c*) XI. Def. 3.
(*d*) I. Ax. 9.

PROPOSITION VI.

THEOREM.—*If two straight lines* (AB, CD) *be at right angles to the same plane,* they shall be parallel to one another.



DEMONSTRATION. Let them meet the plane in the points B, D, *and draw the straight line* BD, *to which draw* DE *at right angles* (*a*), *in the same plane; and make* DE *equal to* AB (*b*), *and join* BE, AE, AD. Then, because AB is perpendi-

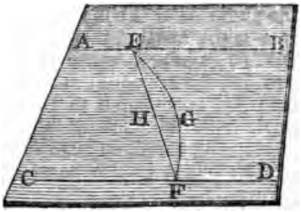
(*a*) I. 11.
(*b*) I. 3.
(*c*) XI. Def 3.
(*d*) I. 4.
(*e*) I. 8.
(*f*) XI. 5.
(*g*) XI. 2.
(*h*) I. 28.

cular to the plane, it shall make right angles with every straight line which meets it, and is in that plane (*c*); but BD, BE, which are in that plane, do each of them meet AB; therefore each of the angles ABD, ABE is a right angle; for the same reason, each of the angles CDB, CDE is a right angle: and because AB is equal to DE, and BD common, the two sides AB, BD are equal to the two ED, DB, each to each; and they contain right angles; therefore the base AD is equal to the base BE (*d*): again, because AB is equal to DE, and BE to AD; AB, BE are equal to ED, DA, each to each; and, in the triangles ABE, EDA, the base AE is common; therefore the angle ABE is equal to the angle EDA (*e*); but ABE is a right angle; therefore EDA is also a right angle, and ED perpendicular to DA: but it is also perpendicular to each of the two BD, DC; wherefore ED is at right angles to each of the three straight lines BD, DA, DC, in the point in which they meet; therefore these three straight lines are all in the same plane (*f*): but AB is in the plane in which are BD, DA, because any three straight lines which meet one another are in one plane (*g*); therefore AB, BD, DC are in one plane: and each of the angles ABD, BDC is a right angle; therefore AB *is parallel to* CD (*h*).

PROPOSITION VII.

THEOREM.—*If two straight lines* (AB, CD) *be parallel,* the straight line drawn from any point (E) in the one to any point (F) in the other, is in the same plane with the parallels.

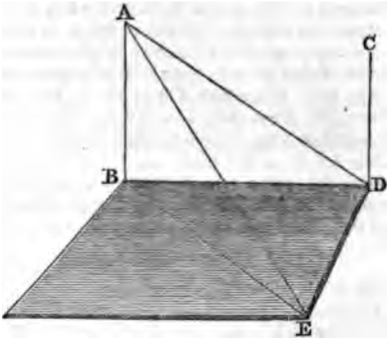
DEMONSTRATION. If not, let it be, if possible, above the plane, as EGF; and in the plane ABCD, in which the parallels are, *draw the straight line* EHF *from* E *to* F: and since EGF also is a straight line, the two straight lines EHF, EGF include a space between them; which is impossible (*a*): therefore *the straight line joining the points* E, F *is not above the plane in which the parallels* AB, CD *are, and is therefore in that plane.*



(*a*) I. Ax. 10.

PROPOSITION VIII.

THEOREM.—*If two straight lines* (AB, CD) *be parallel, and one of them* (AB) *is at right angles to a plane,* the other (CD) shall also be at right angles to the same plane.

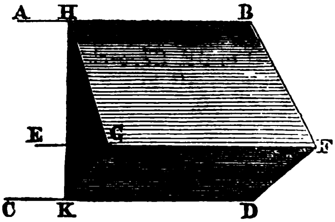


(*a*) XI. 7.
(*b*) I. 11.
(*c*) I. 3.
(*d*) XI. Def. 3.
(*e*) I. 29.
(*f*) I. 4.
(*g*) I. 8.
(*h*) Construction.
(*i*) XI. 4.

DEMONSTRATION. Let AB, CD meet the plane in the points B, D, *and join* BD: therefore AB, CD, BD are in one plane (*a*). *In the plane to which* AB *is at right angles, draw* DE *at right angles to* BD (*b*), *and make* DE *equal to* AB (*c*), *and join* BE, AE, AD. And because AB is perpendicular to the plane, it is perpendicular to every straight line which meets it and is in that plane (*d*); therefore each of the angles ABD, ABE is a right angle: and because the straight line BD meets the parallel straight lines AB, CD, the angles ABD, CDB are together equal to two right angles (*e*): and ABD is a right angle; therefore also CDB is a right angle, and CD perpendicular to BD: and because AB is equal to DE, and BD common, the two AB, BD are equal to the two ED, DB, each to each; and the angle ABD is equal to the angle EDB, because each of them is a right angle; therefore the base AD is equal to the base BE (*f*): again, because AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA, each to each; and the base AE is common to the triangles ABE, EDA; wherefore the angle ABE is equal to the angle EDA (*g*): but ABE is a right angle; and therefore EDA is a right angle, and ED perpendicular to DA: but it is also perpendicular to BD (*h*); therefore ED is perpendicular to the plane which passes through BD, DA (*i*); and therefore makes right angles with every straight line meeting it in that plane (*d*); but DC is in the plane passing through BD, DA, because all three are in the plane in which are the parallels AB, CD; wherefore ED is at right angles to DC; and therefore CD is at right angles to DE: but CD is also at right angles to DB; therefore CD is at right angles to the two straight lines DE, DB, in the point of their intersection D; and therefore *is at right angles to the plane passing through* DE, DB (*i*), *which is the same plane to which* AB *is at right angles.*

PROPOSITION IX.

THEOREM.—*If two straight lines* (AB, CD) *are each of them parallel to the same straight line* (EF), *and not in the same plane with it,* they are parallel to one another.

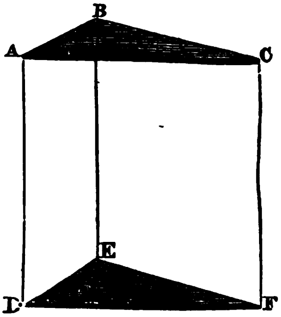


(*a*) I. 11.
(*b*) XI. 4.
(*c*) XI. 8.
(*d*) XI. 6.

DEMONSTRATION. *In* EF, *take any point* G, *from which draw, in the plane passing through* EF, AB, *the straight line* GH *at right angles to* EF (*a*); *and in the plane passing through* EF, CD, *draw* GK *at right angles to the same* EF. And because EF is perpendicular both to GH and GK, EF is perpendicular to the plane HGK passing through them (*b*): and EF is parallel to AB; therefore AB is at right angles to the plane HGK (*c*): for the same reason, CD is likewise at right angles to the plane HGK; therefore AB, CD are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are parallel to one another (*d*); therefore AB *is parallel to* CD.

PROPOSITION X.

THEOREM.—*If two straight lines* (AB, BC) *meeting one another be parallel to two others* (DE, EF) *that meet one another, and are not in the same plane with the first two,* the first two and the other two shall contain equal angles.

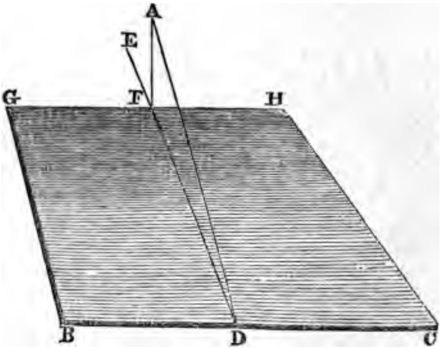


Demonstration. *Take* BA, BC, ED, EF *all equal to one another; and join* AD, CF, BE, AC, DF: then because BA is equal and parallel to ED, therefore AD is both equal and parallel to BE (*a*): for the same reason, CF is equal and parallel to BE; therefore AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, and not in the same plane with it, are parallel to one another (*b*); therefore AD is parallel to CF; and it is equal to it (*c*); and AC, DF join them towards the same parts; and therefore AC is equal and parallel to DF (*a*). And because AB, BC are equal to DE, EF, each to each, and the base AC to the base DF, *the angle* ABC *is equal to the angle* DEF (*d*).

(*a*) I. 33.
(*b*) XI. 9.
(*c*) I. Ax. 1.
(*d*) I. 8.

PROPOSITION XI.

Problem.—To draw a straight line perpendicular to a plane (BH), from a given *point* (A) above it.



Solution. *In the plane, draw any straight line* BC, *and from the point* A, *draw* AD *perpendicular to* BC (*a*): *if then* AD *be also perpendicular to the plane* BH, *the thing required is already done: but if it be not, from the point* D *draw, in the plane* BH, *the straight line* DE *at right angles to* BC (*b*); *and from the point* A, *draw* AF *perpendicular to* DE: AF *shall be perpendicular to the plane* BH.

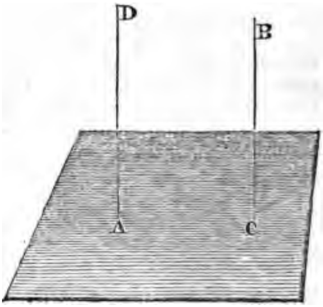
(*a*) I. 12.
(*b*) I. 11.
(*c*) I. 31.
(*d*) XI. 4.
(*e*) XI. 8.
(*f*) XI. Def. 3.

Demonstration. *Through* F, *draw* GH *parallel to* BC (*c*): and because BC is at right angles to ED and DA, BC is at right angles to the plane passing through ED, DA (*d*); and GH is parallel to BC: but, if two straight lines be parallel, one of which is at right angles to a plane, the other is at right angles to the same plane (*e*): wherefore GH is at right angles to the plane through ED, DA; and is perpendicular to every straight line meeting it in that plane (*f*): but AF, which is in the plane through ED, DA, meets it; therefore GH is perpendicular to AF; and consequently AF is perpendicular to GH: and AF is perpendicular to DE; therefore AF is perpendicular to each of the straight lines GH, DE. But if a straight line stand at right angles to each of two straight lines in the point of their intersection, it is also at right angles to the plane passing through them (*d*): but the plane passing through ED, GH, is the plane BH; therefore AF is perpendicular to the plane BH: therefore, *from the given point* A, *above the plane* BH, *the straight line* AF *is drawn perpendicular to that plane.*

PROPOSITION XII.

Problem. To erect a straight line at right angles to a given *plane*, from a *point* (A) given in the plane.

Solution. *From any point* B *above the plane, draw* BC *perpendicular to it* (*a*); *and from* A, *draw* AD *parallel to* BC (*b*).

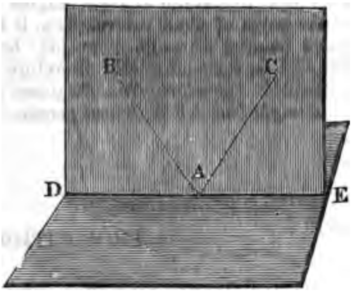


Demonstration. Because, therefore, AD, CB are two parallel straight lines, and one of them BC is at right angles to the given plane, the other AD is also at right angles to it (*c*): therefore, *a straight line has been erected at right angles to a given plane, from a point given in it.*

(*a*) XI. 11.
(*b*) I. 31.
(*c*) XI. 8.

PROPOSITION XIII.

THEOREM.—*From the same point in a given plane*, there cannot be two straight lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular *to a plane from a point above the plane.*



DEMONSTRATION. For, if it be possible, let the two straight lines AB, AC be at right angles to a given plane, from the same point A in the plane, and upon the same side of it. Let a plane pass through BA, AC; the common section of this with the given plane is a straight line passing through A (*a*): let DAE be their common section: therefore the straight lines AB, AC, DAE are in one plane: and because CA is at right angles to the given plane, it makes right angles with every straight line meeting it in that plane (*b*): but DAE, which is in that plane, meets CA; therefore CAE is a right angle: for the same reason, BAE is a right angle; wherefore the angle CAE is equal to the angle BAE (*c*); and they are in one plane, which is impossible. Also, from a point above a plane, there can be but one perpendicular to that plane; for if there could be two, they would be parallel to one another (*d*); which is absurd.

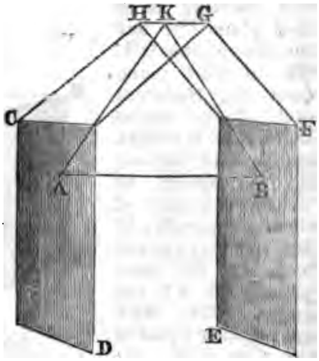
(*a*) XI. 3.
(*b*) XI. Def. 3.
(*c*) XI. Ax.
(*d*) XI. 6.

PROPOSITION XIV.

THEOREM.—*If the same straight line* (AB) *is perpendicular to each of two planes* (CD, EF), they are parallel to one another.

DEMONSTRATION. If not, they shall meet one another when produced: let them meet; their common section is a straight line

GH, in which take any point K, *and join* AK, BK. Then, because AB is perpendicular to the plane EF, it is perpendicular to the straight line BK, which is in that plane (*a*); therefore ABK is a right angle: for the same reason BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK, are equal to two right angles; which is impossible (*b*): therefore the planes CD, EF, though produced, do not meet one another; that is, *they are parallel* (*c*.

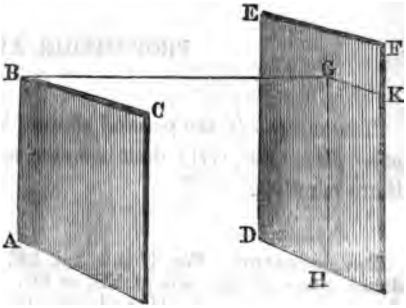


(*a*) XI. Def. 3.
(*b*) I. 17.
(*c*) XI. Def. 8.

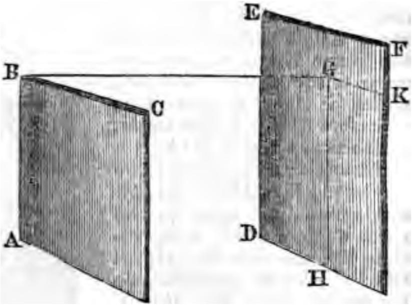
PROPOSITION XV.

THEOREM.—*If two straight lines* (AB, BC) *meeting one another, be parallel to two other straight lines* (DE, EF) *which meet one another, but are not in the same plane with the first two,* the plane which passes through these is parallel to the plane passing through the others.

DEMONSTRATION. *From the point* B, *draw* BG *perpendicular to the plane which passes through* DE, EF (*a*), *and let it meet that plane in* G; *and through* G, *draw* GH *parallel to* ED, *and* GK *parallel to* EF (*b*). And because BG is perpendicular to the plane through DE, EF, it makes right angles with every straight



(*a*) XI. 11. (*b*) I. 81.



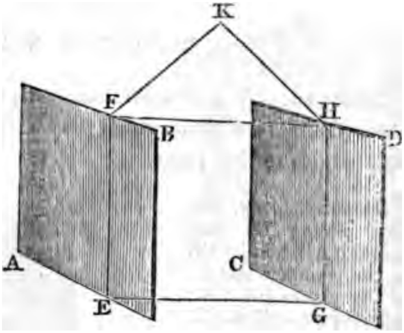
(c) XI. Def. 3.
(d) XI. 9.
(e) I. 29.
(f) XI. 4.
(g) Construction.
(h) XI. 14.

line meeting it in that plane (c): but the straight lines GH, GK in that plane meet it; therefore each of the angles BGH, BGK is a right angle: and because BA is parallel to GH (d) (for each of them is parallel to DE, and they are not both in the same plane with it), the angles GBA, BGH are together equal to two right angles (e): and BGH is a right angle; therefore also GBA is a right angle, and GB perpendicular to BA: for the same reason, GB is perpendicular to BC; since therefore the straight line GB stands at right angles to the two straight lines BA, BC, that cut one another in B; GB is perpendicular to the plane through BA, BC (f): and it is perpendicular to the plane through DE, EF (g); therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF: but planes to which the same straight line is perpendicular, are parallel to one another (h); therefore *the plane through* AB, BC, *is parallel to the plane through* DE, EF.

PROPOSITION XVI.

THEOREM.—*If two parallel planes* (AB, CD) *be cut by another plane* (EF, GH), their common sections (EF, GH), with it are parallels.

DEMONSTRATION. For, if it is not, EF, GH shall meet if produced either on the side of FH, or EG. First, let them be produced on the side of FH, and meet in the point K: therefore, since EFK is in the plane AB, every point in EFK is in that

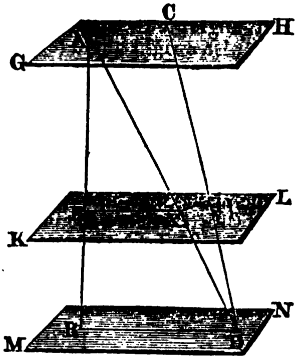


(a) XI. 1.

plane (a): and K is a point in EFK; therefore K is in the plane AB: for the same reason, K is also in the plane CD; wherefore the planes AB, CD, produced, meet one another: but they do not meet, since they are parallel by the hypothesis; therefore the straight lines EF, GH do not meet when produced on the side of FH: in the same manner it may be proved, that EF, GH do not meet when produced on the side of EG. But straight lines which are in the same plane, and do not meet, though produced either way, are parallel; therefore EF *is parallel to* GH.

PROPOSITION XVII.

THEOREM.—*If two straight lines be cut by parallel planes,* they shall be cut in the same ratio.



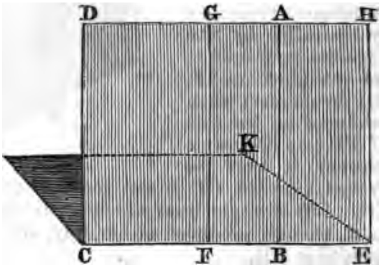
(a) XI. 16. (b) VI. 2.
(c) V. 11.

DEMONSTRATION. Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: as AE is to EB, so shall CF be to FD.

Join AC, BD, AD, *and let* AD *meet the plane* KL *in the point* X; *and join* EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel (a): for the same reason, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel: and because EX is parallel to BD, a side of the triangle ABD; as AE to EB, so is AX to XD (b): again, because XF is parallel to AC, a side of the triangle ADC; as AX to XD, so is CF to FD; and it was proved, that AX is to XD, as AE to EB; therefore, *as* AE *to* EB, *so is* CF *to* FD (c).

ROPOSITION XVIII.

THEOREM.—*If a straight line* (AB) *be at right angles to a plane* (CK), every plane which passes through it shall be at right angles to that plane.

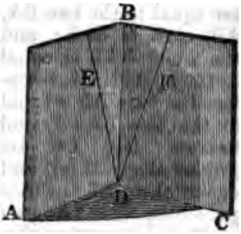


(*a*) I. 11.
(*b*) XI. Def. 3.
(*c*) Construction.
(*d*) I. 28.
(*e*) XI. 8.
(*f*) XI. Def. 4.

DEMONSTRATION. Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; *take any point* F *in* CE, *from which draw* FG, *in the plane* DE, *at right angles to* CE (*a*): and because AB is perpendicular to the plane CK, therefore it is also perpendicular to every straight line in that plane meeting it (*b*), and consequently it is perpendicular to CE; wherefore ABF is a right angle; but GFB is likewise a right angle (*c*); therefore AB is parallel to FG (*d*); and AB is at right angles to the plane CK; therefore FG is also at right angles to the same plane (*e*). But one plane is at right angles to another plane, when the straight lines drawn in one of the planes at right angles to their common section, are also at right angles to the other plane (*f*); and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner it may be proved, *that all planes which pass through* AB, *are at right angles to the plane* CK.

PROPOSITION XIX.

THEOREM.—*If two planes* (AB, CD) *which cut one another be each of them perpendicular to a third plane,* their common section (BD) shall be perpendicular to the same plane.

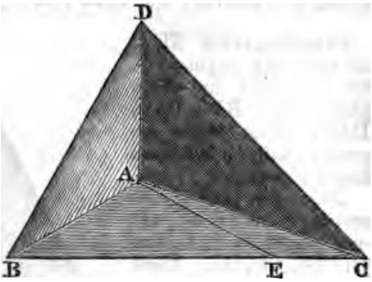


(a) I. 11.
(b) XI. Def. 4.
(c) XI. 13.

DEMONSTRATION. If it be not, *from the point* D, *draw in the plane* AB, *the straight line* DE *at right angles to* AD (*a*), the common section of the plane AB with the third plane; *and in the plane* BC, *draw* DF *at right angles to* CD, the common section of the plane BC with the third plane. And because the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common section, DE is perpendicular to the third plane (*b*): in the same manner it may be proved, that DF is perpendicular to the third plane; wherefore, from the point D, two straight lines stand at right angles to the third plane, upon the same side of it; which is impossible (*c*): therefore, from the point D, there cannot be any straight line at right angles to the third plane, except BD the common section of the planes AB, BC; therefore BD *is perpendicular to the third plane.*

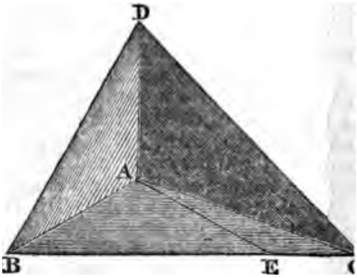
PROPOSITION XX.

THEOREM.—*If a solid angle* (A) *be contained by three plane angles* (BAC, CAD, DAB), any two of them are greater than the third.



(a) I 23.

DEMONSTRATION. If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third: but if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; *and at the point* A, *in the straight line* AB, *make in the plane which passes through* BA, AC, *the angle* BAE *equal to the angle* DAB (*a*); *and make* AE *equal to* AD, *and through* E, *draw* BEC, *cutting* AB, AC *in the points* B, C, *and join* DB, DC. And because DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, each to each; and the angle DAB is equal to the angle EAB; therefore the base DB is equal to the base BE (*b*): and because BD, DC are greater than CB (*c*), and one of them BD has been proved equal to BE a part of CB, therefore the other DC is greater than the remaining part EC (*d*): and because DA is equal to AE, and AC common, but the base DC greater than the base EC; therefore the angle DAC is greater than the angle EAC (*e*): and, by the construction, the angle DAB is equal to tl angle BAE; wherefore the angles DAB, DAC are together great than BAE, EAC (*f*), that is, than the angle BAC: but BAC is n less than either of the angles DAB, DAC; therefore BAC *wi either of them is greater than the other.*



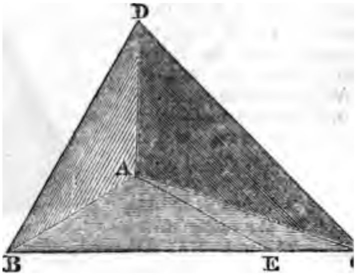
(*b*) I. 4.
(*c*) I. 20.
(*d*) I. Ax. 5.
(*e*) I. 25.
(*f*) I. Ax. 4.

PROPOSITION XXI.

THEOREM.—*Every solid angle* is contained by plane angle which together are less than four right angles.

DEMONSTRATION. First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB: these three together shall be less than four right angles.

Take, in each of the straight lines AB, AC, AD, *any points* B, C, D, *and join* BC, CD, DB. Then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater than the third (*a*); therefore the angles CBA, ABD are greater than tl

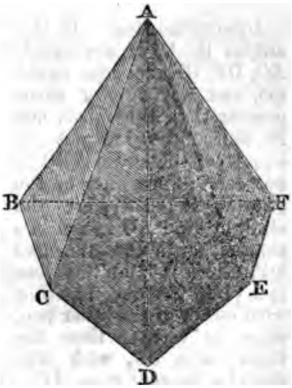


(*a*) XI. 20.
(*b*) I. 32.

angle DBC: for the same reason, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC; wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, DCB, BDC: but the three angles DBC, DCB, BDC are equal to two right angles (*b*): therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles: and because the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of these three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD are equal to six right angles: of these, the six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles; therefore *the remaining three angles* BAC, DAC, BAD, *which contain the solid angle at* A, *are less than four right angles.*

Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB: these shall together be less than four right angles.

Let the planes in which the angles are be cut by a plane, and let the common sections of it with those planes be BC, CD, DE, EF, FB. And because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater than the third (*a*), the angles CBA, ABF are greater than the angle FBC: for the same reason, the two plane angles at each of the points C, D, E, F, viz. those angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon BCDEFB; therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon: and because all the angles of the triangles are together equal to twice as many right angles as there are triangles (*b*), that is, as there are sides in the polygon BCDEFB; and that all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon (*c*); therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles (*d*): but all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved; wherefore *the remaining angles of the triangles,* viz.



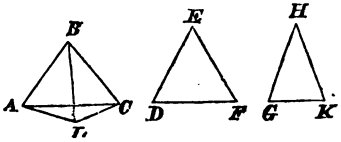
(*c*) I. 32 B, cor. 7.
(*d*) I. Ax. 1.

those at the vertex, which contain the solid angle at A, *are less than four right angles.*

Scholium. This proposition does not hold good if any of the angles of the rectilineal figure BCDEFB be *re-entrant*, the reason of which will be seen from the Scholia to Corollaries 7 and 8, Prop. 32 B, Book 1.

PROPOSITION XXII.

Theorem.—*If every two of three plane angles* (B, E, H) *be greater than the third, and if the straight lines* (AB, BC, DE, EF, GH, HK) *which contain them be all equal,* a triangle may be made of the straight lines (AC, DF, GK) that join the extremities of those equal straight lines.



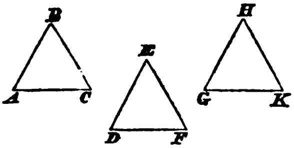
(*a*) I. 4.
(*b*) I. 4 or 24.
(*c*) I. 23.
(*d*) Hypoth.
(*e*) I. Ax. 5.
(*f*) I. 24.
(*g*) I. Ax. 4.
(*h*) I. 20.
(*i*) I. 22.

Demonstration. If the angles B, E, H are equal, AC, DF, GK are also equal (*a*), and any two of them greater than the third: but if the angles are not all equal, let the angle ABC be not less than either of the two E, H; therefore the straight line AC is not less than either of the other two DF, GK (*b*): and therefore it is plain that AC, together with either of the other two, must be greater than the third: also DF, with GK, shall be greater than AC; *for at the point* B, *in the straight line* AB, *form the angle* ABL *equal to the angle* H (*c*), *and make* BL *equal to one of the straight lines* AB, BC, DE, EF, GH, HK, *and join* AL, LC. Then, because AB, BL are equal to GH, HK, each to each, and the angle ABL to the angle GHK, the base AL is equal to the base GK (*a*): and because the angles E, H are greater than the angle ABC (*d*), of which the angle H is equal to ABL, therefore the remaining angle E is greater than the angle LBC (*e*): and because the two sides LB, BC are equal to the two DE, EF, each to each, and that the angle E is greater than the angle LBC, the base DF is greater than the base LC (*f*): and it has been proved that GK is equal to AL; therefore DF and GK are greater than AL and LC (*g*): but AL and LC are greater than AC (*h*); much more than are DF and GK greater than AC. Wherefore, every two of these straight lines AC, DF, GK are greater than the third; and, therefore, *a triangle may be made* (*i*), *the sides of which shall be equal to* AC, DF, GK.

PROPOSITION XXIII.

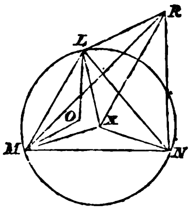
Problem. To make a solid angle which shall be contained by three given *plane angles* (B, E, H), any two of them being greater than the third, and all three together less than four right angles.

Solution. *From the straight lines which contain the angles, cut off* AB, BC, DE, EF, GH, HK, *all equal to one another; and join* AC, DF, GK: *then a triangle may be made of three straight lines equal to* AC, DF, GK (*a*): *let this be the triangle* LMN, *so that* AC *be equal to* LM, DF *to* MN, *and* GK *to* LN (*b*); *and about the triangle* LMN *describe a circle* (*c*), *and find its center* X (*d*), *which will be either within the triangle, or in one of its sides, or without it.*



(*a*) XI. 22.
(*b*) I. 22.
(*c*) VI. 5.
(*d*) III. 1.
(*e*) I. 8.

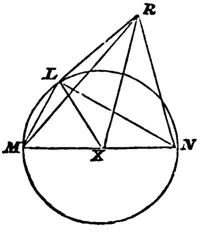
First, let the center X *be within the triangle, and join* LX, MX, NX: AB *shall be greater than* LX. *If not,* AB *must either be equal to, or less than* LX: *first let it be equal: then, because* AB *is equal to* LX, *and that* AB *is also equal to* BC, *and* LX *to* XM, AB *and* BC *are equal to* LX *and* XM, *each to each; and the base* AC *is, by construction, equal to the base* LM; *wherefore the angle* B *is equal to the angle* LXM (*e*): *for the same reason, the angle* E *is equal to the angle* MXN, *and the angle* H *to the angle* NXL; *therefore the three angles* B, E, H *are equal to the three angles* LXM, MXN, NXL: *but the three angles* LXM, MXN, NXL *are equal to four right angles* (*f*); *therefore also the three angles* B, E, H *are equal to four right angles: but, by the hypothesis, they are less than four right angles; which is absurd: therefore* AB *is not equal to* LX. *But neither can* AB *be less than* LX: *for, if possible, let it be less: and upon the straight line* LM, *on the side of it on which is the center* X, *describe the triangle* LOM (*b*),



(*f*) I. 13, Cor. 3.
(*g*) I. 21.

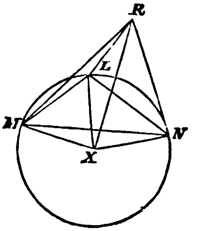
of which, two of the sides LO, OM *are equal to* AB, BC: *and because the base* LM *is equal to the base* AC, *the angle* O *is equal to the angle* B (*e*): *and* AB, *that is* LO, *is, by the hypothesis, less than* LX: *wherefore* LO, OM *fall within the triangle* LXM; *for, if they fell upon its sides, or without it, they would be equal to, or greater than,* LX, XM (*g*); *therefore the angle* O, *that is, the angle* B, *is greater than the angle* LXM (*g*): *in the same manner it may be proved, that the angle* E *is greater than the angle* MXN, *and the angle* H *greater than the angle* NXL; *therefore the three angles* B, E, H *are greater than the three angles* LXM, MXN, NXL, *that is, than four right angles* (*f*): *but the same angles* B, E, H *are less than four right angles* (*h*); *which is absurd; therefore* AB *is not less than* LX: *and it has been proved, that it is not equal to* LX; *wherefore* AB *is greater than* LX.

Next, let the center X *of the circle fall in one of the sides of the triangle, viz. in* MN, *and join* XL: *in this case also,* AB *shall be greater than* LX; *if not,* AB *is either equal to* LX, *or less than it. First, let it be equal to* LX; *therefore* AB *and* BC, *that is,* DE *and* EF, *are equal to* MX *and* XL, *that is, to* MN: *but, by the construction,* MN *is equal to* DF; *therefore* DE, EF *are equal to* DF; *which is impossible* (*i*); *wherefore* AB *is not equal to* LX: *nor is it less; for then, much more, an absurdity would follow; therefore* AB *is greater than* LX.

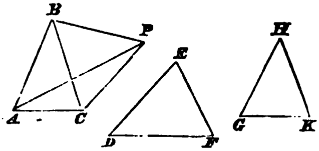


(*h*) Hypoth.
(*i*) I. 20.
(*k*) I. 32.
(*l*) I. 24.
(*m*) I. 25.
(*n*) XI. 12.

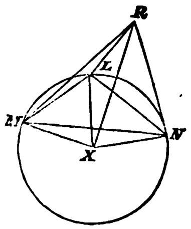
But let the center X *of the circle fall without the triangle* LMN, *and join* LX, MX, NX: *in this case, likewise,* AB *shall be greater than* LX; *if not, it is either equal to or less than* LX. *First, let it be equal: it may be proved, in the same manner as in the first case, that the angle* B *is equal to the angle* MXL, *and* H *to* LXN; *therefore the whole angle* MXN *is equal to the two angles* B, H: *but* B *and* H *are together greater than the angle* E (*h*): *therefore also the angle* MXN *is greater than* E: *and because* DE, EF *are equal to* MX, XN, *each to each, and the base* DF *to the base* MN, *the angle* MXN *is equal to the angle* E (*e*): *but it has been proved, that it is greater than* E; *which is absurd; therefore* AB *is not equal to* LX: *neither is it less; for then, as has been proved in the first case, the angle* B *is greater than the angle* MXL, *and the angle* H *greater than the angle* LXN. *At the point* B, *in the straight line* CB, *make the angle* CBP *equal to the angle* H, *and make* BP



equal to HK, *and join* CP, AP. *And because* CB *is equal to* GH, CB, BP *are equal to* GH, HK, *each to each; and they contain equal angles; wherefore the base* CP *is equal to the base* GK, *that is, to* LN. *And in the isosceles triangles* ABC, MXL, *because the angle* ABC *is greater than the angle* MXL, *therefore the angle* MLX *at the base is greater than the angle* ACB *at the base* (*k*): *for the same reason, because the angle* H *or* CBP, *is greater than the angle* LXN, *the angle* XLN *is greater than the angle* BCP; *therefore the whole angle* MLN *is greater than the whole angle* ACP. *And because* ML, LN *are equal to* AC, CP, *each to each, but the angle* MLN *is greater than the angle* ACP, *the base* MN *is greater than the base* AP (*l*); *but* MN *is equal to* DF; *therefore also* DF *is greater than* AP. *Again, because* DE, EF *are equal to* AB, BP, *each to each, but the base* DF *greater than the base* AP, *the angle* E *is greater than the angle* ABP (*m*): *but* ABP *is equal to the two angles* ABC, CBP, *that is, to the two angles* ABC, H: *therefore the angle* E *is greater than the two angles* ABC, H: *but it is also less than these* (*h*); *which is impossible; therefore* AB *is not less than* LX: *and it has been proved, that it is not equal to it; therefore* AB *is greater than* LX.



From the point X, *erect* XR *at right angles to the plane of the circle* LMN (*n*). *And because it has been proved in all the cases, that* AB *is greater than* LX, *find a square equal to the excess of the square on* AB *above the square on* LX, *and make* RX *equal to its side, and join* RL, RM, RN: *the solid angle at* R *shall be the angle required.*



DEMONSTRATION. Because RX is perpendicular to the plane of the circle LMN, it is perpendicular to each of the straight lines LX, MX, NX (*o*). And because LX is equal to MX, and XR common, and at right angles to each of them, the base RL is equal to the base RM (*p*): for the same reason, RN is equal to each of the two RL, RM; therefore the three straight lines RL, RM, RN are all equal. And because the square on XR is equal to the excess of the square on AB above the square on LX; therefore the square on AB is equal to the squares on LX, XR: but the square on RL is equal to the same squares, because LXR is a right angle (*q*); therefore the square on AB is equal to the square on RL, and the straight line AB to RL. But each of the straight

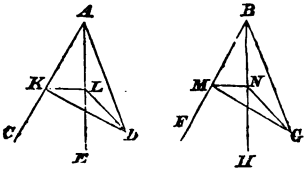
(*o*) XI. Def. 3.
(*p*) I. 4.
(*q*) I. 47.

lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL; therefore AB, BC, DE, EF, GH, HK are each of them equal to each of the straight lines RL, RM, RN. And because RL, RM are equal to AB, BC, each to each, and the base LM to the base AC, the angle LRM is equal to the angle B (*e*): for the same reason, the angle MRN is equal to the angle E, and NRL to H. Therefore, *there is made a solid angle at* R, *which is contained by three plane angles* LRM, MRN, NRL, *which are equal to the three given plane angles* B, E, H, *each to each.*

PROPOSITION A.

THEOREM.—*If each of two solid angles be contained by three plane angles, which are equal to one another, each to each,* the planes in which the equal angles are, have the same inclination to one another.

DEMONSTRATION. Let there be two solid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG: the planes in which the equal angles are shall have the same inclination to one another.



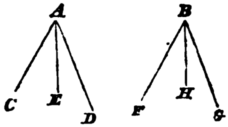
(*a*) I. 11.
(*b*) XI. Def. 6.
(*c*) Hypoth.
(*d*) I. 26.
(*e*) I. 4.
(*f*) I. 8.
(*g*) XI. Def. 3.

In the straight line AC, *take any point* K, *and from* K *draw, in the plane* CAD, *the straight line* KD *at right angles to* AC (*a*), *and in the plane* CAE, *the straight line* KL *at right angles to the same* AC: therefore the angle DKL is the inclination of the plane CAD to the plane CAE (*b*). *In* BF, *take* BM *equal to* AK, *and from the point* M, *draw in the planes* FBG, FBH, *the straight lines* MG, MN *at right angles to* BF; therefore the angle GMN is the inclination of the plane FBG to the plane FBH (*b*). *Join* LD, NG. And because in the triangles KAD, MBG, the angles KAD, MBG are equal (*c*), as also the right angles AKD, BMG, and that the sides AK, BM, adjacent to the equal angles, are equal to one another, therefore KD is equal to MG (*d*), and AD to BG: for the same reason, in the triangles KAL, MBN, KL is equal to MN, and AL to BN; therefore in the triangles LAD, NBG, LA, AD are equal to NB, BG, each to each; and they contain equal angles; therefore the base LD is equal to the base NG (*e*). Lastly, in the triangles KLD, MNG, the sides DK, KL are equal to GM, MN, each to each, and the base LD to the base NG; therefore the angle DKL is equal to the angle GMN (*f*): but the angle DKL is the inclination of the plane CAD to the plane CAE, and the angle GMN is the inclination of the plane FBG to the plane FBH, which planes have therefore the same inclination to one another (*g*). *And in the same manner it may be demonstrated, that the other planes in which the equal angles are, have the same inclination to one another.*

PROPOSITION B.

THEOREM.—*If two solid angles be contained, each by three plane angles which are equal to one another, each to each, and alike situated,* these solid angles are equal to one another.

DEMONSTRATION. Let there be two solid angles at A and B, of which the solid angle at A is contained by the three plane angles CAD, CAE, EAD; and that at B by the three plane angles FBG, FBH, HBG; of which CAD is equal to FBG; CAE to FBH; and EAD to HBG: the solid angle at A shall be equal to the solid angle at B.



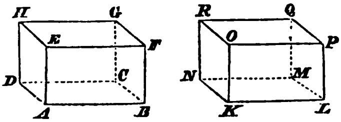
(*a*) XI. A.
(*b*) I. Ax. 8.

Let the solid angle at A be applied to the solid angle at B: and first, the plane angle CAD being applied to the plane angle FBG, so that the point A may coincide with the point B, and the straight line AC with BF; then AD coincides with BG, because the angle CAD is equal to the angle FBG: and because the inclination of the plane CAE to the plane CAD, is equal (*a*) to the inclination of the plane FBH to the plane FBG, the plane CAE coincides with the plane FBH, because the planes CAD, FBG coincide with one another: and because the straight lines AC, BF coincide, and that the angle CAE is equal to the angle FBH; therefore AE coincides with BH: and AD coincides with BG; wherefore the plane EAD coincides with the plane HBG: therefore, the solid angle A coincides with the solid angle B, *and consequently they are equal to one another* (*b*).

PROPOSITION C.

THEOREM.—*Solid figures which are contained by the same number of equal and similar planes alike situated, and having none of their solid angles contained by more than three plane angles*, are equal and similar to one another.

DEMONSTRATION. Let AG, KQ be two solid figures contained by the same number of similar and equal planes, alike situated, viz. let the plane AC be similar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and, lastly, FH similar and equal to PR; the solid figure AG shall be equal and similar to the solid figure KQ.



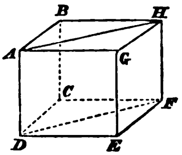
(*a*) XI. B.

Because the solid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypothesis, are equal to the plane angles LKN, LKO, OKN, which contain the solid angle at K, each to each, therefore the solid angle at A is equal to the solid angle at K (*a*): in the same manner, the other solid angles of the figures are equal to one another. Let, then, the solid figure AG be applied to the solid figure KQ: first, the plane figure AC being applied to the plane figure KM, so that the straight line AB may coincide with KL, the figure AC must coincide with the figure KM, because they are equal and similar; therefore the straight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B with the points K, N, M, L: and the solid angle at A coincides with the solid angle at K (*a*): wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP, because they are equal and similar to one another: therefore the straight lines AE, EF, FB coincide with KO, OP, PL; and the points E, F, with the points O, P: in the same manner, the figure AH coincides with the figure KR, and the straight line DH, with NR, and the point H with the point R. And because the solid angle at B is equal to the solid angle at L, it may be proved in the same manner, that the figure BG coincides with the figure LQ, and the straight line CG with MQ, and the point G with the point Q. Therefore, since all the planes and sides of the solid figure AG coincide with the planes and sides of the solid figure KQ, AG *is equal and similar to* KQ. And in the same manner, any other solid figures whatever, contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, *may be proved to be equal and similar to one another.*

PROPOSITION XXIV.

THEOREM.—*If a solid be contained by six planes, two and two of which are parallel;* the opposite planes are similar and equal parallelograms.

DEMONSTRATION. Let the solid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE: its opposite planes shall be similar and equal parallelograms.



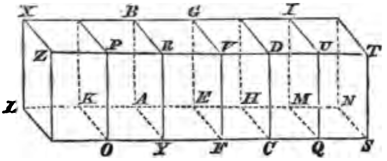
(*a*) XI. 16.
(*b*) XI. 10.
(*c*) I. 4.
(*d*) I. 34.

Because the two parallel planes BG, CE are cut by the plane AC, their common sections AB, CD are parallel (*a*): again, because the two parallel planes BF, AE are cut by the plane AC, their common sections AD, BC are parallel (*a*): and AB is parallel to CD; therefore AC is a parallelogram. In like manner it may be proved, that each of the figures CE, FG, GB, BF, AE is a parallelogram. *Join* AH, DF: and because AB is parallel to DC, and BH to CF; the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another, and are not in the same plane with the other two: wherefore they contain equal angles (*b*); therefore the angle ABH is equal to the angle DCF: and because AB, BH are equal to DC, CF, each to each, and the angle ABH equal to the angle DCF; therefore the base AH is equal to the base DF (*c*), and the triangle ABH to the triangle DCF: but the parallelogram BG is double of the triangle ABH (*d*), and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner it may be proved, *that the parallelogram* AC *is equal and similar to the parallelogram* GF, *and the parallelogram* AE *to* BF.

PROPOSITION XXV.

THEOREM.—*If a solid parallelopiped be cut by a plane parallel to two of its opposite planes,* it divides the whole into two solids, the base of one of which shall be to the base of the other, as the one solid is to the other.

Demonstration. Let the solid parallelopided ABCD be cut by the plane EV, which is parallel to the opposite planes, AR, HD, and divides the whole into the two solids ABFV, EGCD: as the base AEFY of the first is to the base EHCF of the other, so shall the solid ABFV be to the solid EGCD.



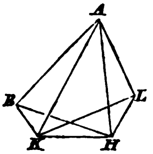
(*a*) I. 36.
(*b*) XI. 24.
(*c*) XI. c.
(*d*) V. Def. 5.

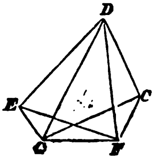
Produce AH, *both ways, and take any number of straight lines* HM, MN, *each equal to* EH, *and any number* AK, KL, *each equal to* EA, *and complete the parallelograms* LO, KY, HQ, MS, *and the solids* LP, KR, HU, MT. Then, because the straight lines LK, KA, AE are all equal, the parallelograms LO, KY, AF are equal (*a*); and likewise the parallelograms KX, KB, AG: also the parallelograms LZ, KP, AR are equal, because they are opposite planes (*b*); for the same reason, the parallelograms EC, HQ, MS are equal (*a*), and the parallelograms HG, HI, IN: as also HD, MU, NT (*b*): therefore three planes of the solid LP are equal and similar to three planes of the solid KR, as also to three planes of the solid AV: but the three planes opposite to these three are equal and similar to them in the several solids (*b*), and none of their solid angles are contained by more than three plane angles; therefore the three solids LP, KR, AV are equal to one another (*c*): for the same reason, the three solids ED, HU, MT are equal to one another: therefore what multiple soever the base LF is of the base AF, the same multiple is the solid LV of the solid AV; and whatever multiple the base NF is of the base HF, the same multiple is the solid NV of the solid ED; and if the base LF be equal to the base NF, the solid LV is equal to the solid NV (*c*); and if the base LF be greater than the base NF, the solid LV is greater than the solid NV; and if less, less. Since then there are four magnitudes, viz. the two bases AF, FH, and the two solids AV, ED; and that of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever; and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever; and since it has been proved, that if the base LF is greater than the base FN, the solid LV is greater than the solid NV; and if equal, equal; and if less, less; therefore *as the base* AF *is to the base* FH, *so is the solid* AV *to the solid* ED (*d*).

PROPOSITION XXVI.

PROBLEM. At a given *point* (A) in a given *straight line* (AB) to make a solid angle equal to a given *solid angle* (D) *contained by three plane angles* (EDC, EDF, FDC).

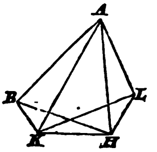
SOLUTION. *In the straight line* DF *take any point* F, *from which draw* FG *perpendicular to the plane* EDC (*a*), *meeting that plane in* G, *and join* DG: *at the point* A, *in the straight line* AB, *form the angle* BAL *equal to the angle* EDC (*b*); *and in the plane* BAL, *form the angle* BAK *equal to the angle* EDG; *then make* AK *equal to* DG, *and from the point* K, *erect* KH *at right angles to the plane* BAL (*c*), *and make* KH *equal to* GF, *and join* AH: *the solid angle at* A *which is contained by the three plane angles* BAL, BAH, HAL, *shall be equal to the solid angle at* D *contained by the three plane angles* EDC, EDF, FDG.



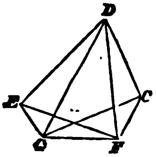


(*a*) XI. 11.
(*b*) I. 23.
(*c*) XI. 12.
(*d*) XI. Def. 3.
(*e*) I. 4.
(*f*) Constr.
(*g*) I. 8.

DEMONSTRATION. *Take the equal straight lines* AB, DE, *and join* HB, KB, FE, GE. And because FG is perpendicular to the plane EDC, it makes right angles with every straight line meeting it in that plane (*d*); therefore each of the angles FGD, FGE is a right angle: for the same reason, HKA, HKB are right angles. And because KA, AB are equal to GD, DE, each to each, and that they contain equal angles, therefore the base BK is equal to the base EG (*e*); and KH is equal to GF (*f*), and HKB, FGE are right angles, therefore HB is equal to FE (*e*). Again, because AK, KH are equal to DG, GF, each to each, and contain right angles, the base AH is equal to the base DF; and AB is equal to DE; therefore, HA, AB are equal to FD, DE, each to each; and the base HB is equal to the base FE; therefore the angle BAH is equal to the angle EDF (*g*): for the same reason, the angle HAL is equal to the angle FDC: because if AL and DC be made equal, and KL, HL, GC, FC be joined; since the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the construction, equal, therefore the remaining angle KAL is equal to the remaining angle GDC: and because KA, AL are equal to GD, DC, each to each, and contain equal angles, the base KL is equal to the base GC (*e*); and KH is equal to GF; so that LK, KH are equal to CG, GF, each to each; and they contain right angles (*d*), therefore the base HL is equal to the base FC (*e*): again, because HA, AL are equal to FD, DC, each to each, and the base HL to the base FC, the angle HAL is equal to the angle FDC (*g*). Therefore, because the three plane angles BAL, BAH, HAL, which contain the solid angle at A, are equal to the three plane angles EDC, EDF, FDC, which contain the solid angle at D, each to each, and are situated in the same order, the solid angle at A is equal to the solid angle at D (*h*). Therefore *at a given point in a given straight line, a solid angle has been made equal to a given solid angle contained by three plane angles.*



(*d*) XI. Def. 3.
(*e*) I. 4.

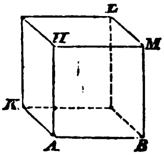


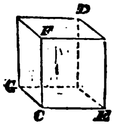
(*g*) I. 8.
(*h*) XI. B.

PROPOSITION XXVII.

Problem. To describe, from a given *straight line* (AB), a solid parallelopiped similar and similarly situated *to one* given (CD).

Solution. *At the point* A *of the given straight line* AB, *form a solid angle equal to the solid angle at* C (*a*), *and let* BAK, KAH, HAB *be the three plane angles which contain it, so that* BAK *be equal to the angle* ECG, *and* KAH *to* GCF, *and* HAB *to* FCE: *and as* EC *is to* CG, *so make* BA *to* AK (*b*); *and as* GC *is to* CF, *so make* KA *to* AH (*b*); *wherefore, ex æquali, as* EC *is to* CF, *so is* BA *to* AH (*c*): *complete the parallelogram* BH, *and the solid* AL: AL *shall be similar and similarly situated to* CD.



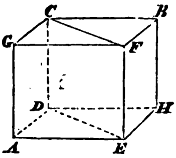


(*a*) XI. 26.
(*b*) VI. 12.
(*c*) V. 22.
(*d*) VI. Def. 1.
(*e*) XI. 24.
(*f*) XI. B.
(*g*) XI. Def. 11.

Demonstration. Because, as EC is to GC, so BA to AK, the sides about the equal angles ECG, BAK, are proportionals; therefore the parallelogram BK is similar to EG (*d*): for the same reason, the parallelogram KH is similar to GF, and HB to FE; wherefore three parallelograms of the solid AL are similar to three of the solid CD: and the three opposite ones in each solid are equal and similar to these, each to each (*e*). Also, because the plane angles which contain the solid angles of the figures are equal, each to each, and situated in the same order, the solid angles are equal, each to each (*f*): therefore the solid AL is similar to the solid CD (*g*). Wherefore, *from a given straight line* AB, *a solid parallelopiped* AL *has been described similar and similarly situated to the given one* CD.

PROPOSITION XXVIII.

THEOREM.—*If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes*, it shall be cut into two equal parts.



DEMONSTRATION. Let AB be a solid parallelopiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz. those which are drawn betwixt the equal angles in each: and because CD, FE are each of them parallel to GA, and not in the same plane with it, CD, FE are parallel (*a*): wherefore the diagonals CF, DE are in the plane in which the parallels are, and are themselves parallels (*b*): and the plane CDEF shall cut the solid AB into two equal parts.

Because the triangle CGF is equal to the triangle CBF (*c*), and the triangle DAE to DHE; and that the parallelogram CA is equal and similar to the opposite one BE (*d*); and the parallelogram GE to CH; therefore the prism contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal to the prism contained by the two triangles CBF, DHE (*e*), and the three parallelograms BE, CH, EC; because they are contained by the same number of equal and similar planes, alike situated, and none of their solid angles are contained by more than three plane angles. Therefore, *the solid* AB *is cut into two equal parts by the plane* CDEF.

(*a*) XI. 9.
(*b*) XI. 16.
(*c*) I. 34.
(*d*) XI. 24.
(*e*) XI. c.

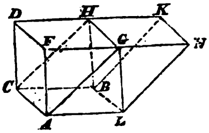
SCHOLIUM. The *insisting* straight lines of a parallelopiped, mentioned in some of the following propositions, are the sides of the parallelograms betwixt the base and the opposite plane parallel to it.

PROPOSITION XXIX.

THEOREM.—*If solid parallelopipeds are upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base,* they are equal to one another.

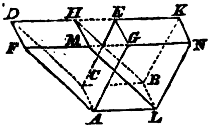
DEMONSTRATION. Let the solid parallelopipeds AH, AK be upon the same base AB, and of the same altitude, and let their insisting straight lines AF, AG, LM, LN be terminated in the same straight line FN, and CD, CE, BH, BK be terminated in the same straight line DK: the solid AH shall be equal to the solid AK.

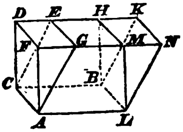
First, let the parallelograms DG, HN, which are opposite to the base AB, have a common side HG. Then because the solid AH is cut by the plane AGHC passing through the diagonals AG, CH of the opposite planes ALGF, CBHD, AH is cut into two equal parts by the plane AGHC (*a*); therefore the solid AH is double of the prism which is contained betwixt the triangles ALG, CBH: for the same reason, because the solid AK is cut by the plane LGHB, through the diagonals LG, BH of the opposite planes ALNG, CBKH, the solid AK is double of the same prism which is contained betwixt the triangles ALG, CBH: therefore *the solid* AH *is equal to the solid* AK (*b*).



(*a*) XI. 28.
(*b*) I. Ax. 6.
(*c*) I. 34.
(*d*) I. Ax. 2 or 3.
(*e*) I. 38.
(*f*) I. 36.
(*g*) XI. 24.
(*h*) XI. c.
(*i*) I. Ax. 3.

Next, let the parallelograms DM, EN, opposite to the base, have no common side. Then, because CH, CK are parallelograms, CB is equal to each of the opposite sides DH, EK (*c*): wherefore DH is equal to EK: add, or take away the common part HE; then DE is equal to HK (*d*); wherefore also the triangle CDE is equal to the triangle BHK (*e*), and the parallelogram DG is equal to the parallelogram HN (*f*): for the same reason, the triangle AFG is equal to the triangle LMN: and the parallelogram CF is equal to the parallelogram BM, and CG to



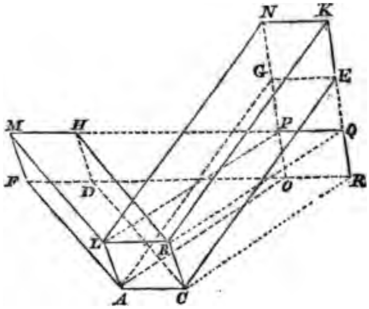


BN (*g*); for they are opposite. Therefore the prism which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC is equal (*h*) to the prism contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. If, therefore, the prism LMN, BHK be taken from the solid of which the base is the parallelogram AB, and in which FDKN is the one opposite to it; and if from this same solid there be taken the prism AFG, CDE; the remaining solid, viz. *the parallelopiped* AH *is equal to the remaining parallelopiped* AK (*i*).

PROPOSITION XXX.

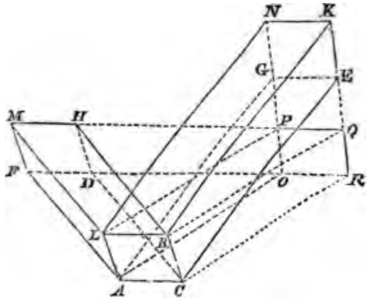
THEOREM.—*If solid parallelopipeds are upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base,* they are equal to one another.

DEMONSTRATION. Let the parallelopipeds CM, CN be upon the same base AB, and of the same altitude, but their insisting straight lines AF, AG, LM, LN, CD, CE, BH, BK not terminated in the same straight lines: the solids CM, CN shall be equal to one another.



(*a*) Hypoth.

Produce FD, MH, *and* NG, KE, *and let them meet one another in the points* O, P, Q, R; *and join* AO, LP, BQ, CR. And because the plane LBHM is parallel to the opposite plane ACDF, and that the plane LBHM is that in which are the parallels LB, MHPQ, in which also is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, in which also is the figure CAOR; therefore the figures BLPQ, CAOR, are in parallel planes: in like manner, because the plane ALNG is parallel to the opposite plane CBKE, and that the plane ALNG is that in which are the parallels AL, OPGN, in which also is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, in which also is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes: and the planes ACBL, ORQP are parallel (*a*); therefore the solid CP is a parallelopiped: but the solid CM is equal to the solid CP (*b*), because they are upon the same base ACBL, and their insisting straight lines AF, AO, CD, CR, LM, LP, BH, BQ are in the same straight lines FR, MQ; and the solid CP is equal to the solid CN (*b*), for they are upon the same base ACBL, and their insisting straight lines AO, AG, LP, LN, CR, CE, BQ, BK are in the same straight lines ON, RK; therefore *the solid* CM *is equal to the solid* CN.

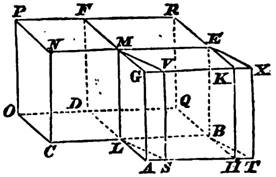


(*b*) XI. 29.

PROPOSITION XXXI.

THEOREM.—*If solid parallelopipeds* (AE, CF) *are upon equal bases* (AB, CD), *and of the same altitude,* they are equal to one another.

DEMONSTRATION. First, let the insisting straight lines be at right angles to the bases AB, CD, and let the bases be placed in the same plane, and so that the sides CL, LB may be in a straight line; therefore the straight line LM, which is at right angles to the plane in which the bases are, in the point L, is common to the two solids AE, CF (*a*): let the other insisting lines of the solids be AG, HK, BE; DF, OP, CN: and first, let the angle ALB be equal to the angle CLD: then AL, LD are in a straight line (*b*). Produce OD, HB, and let them meet in Q, and complete the solid parallelopiped LR, the base of which is the parallelogram LQ, and of which LM is one of its insisting straight lines. Therefore,

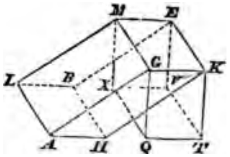


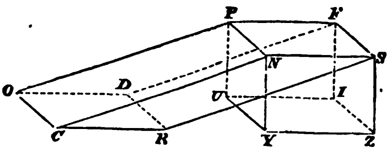
(*a*) XI. 13.
(*b*) I. 14.
(*c*) V. 7.
(*d*) XI. 25.
(*e*) V. 11.
(*f*) V. 9.
(*g*) XI. 29.
(*h*) I. 35.
(*i*) XI. 11.

because the parallelogram AB is equal to CD, as the base AB is to the base LQ, so is the base CD to the base LQ (*c*). And because the solid parallelopiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is the solid AE to the solid LR (*d*): for the same reason, because the solid parallelopiped CR is cut by the plane LMFD, which is parallel to the opposite planes CP, BR; as the base CD is to the base LQ, so is the solid CF to the solid LR: but as the base AB is to the base LQ, so the base CD to the base LQ, as before was proved: therefore, as the solid AE to the solid LR, so is the solid CF to the solid LR (*e*): and therefore *the solid* AE *is equal to the solid* CF (*f*).

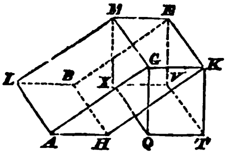
But let the solid parallelopipeds SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases; and place the bases SB, CD in the same plane, so that CL, LB may be in a straight line; and let the angles SLB, CLD be unequal: the solid SE shall be equal to the solid CF. Produce DL, TS until they meet in A; and from B, draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the solids AE, LR: therefore the solid AE is equal to the solid SE (*g*), because they are upon the same base LE, and of the same altitude, and their insisting straight lines, viz. LA, LS, BH, BT, MG, MV, EK, EX, are in the same straight lines AT, GX: and because the parallelogram AB is equal to SB (*h*), for they are upon the same base LB, and between the same parallels LB, AT: and that the base SB is equal to the base CD; therefore the base AB is equal to the base CD; and the angle ALB is equal to the angle CLD; therefore by the first case, the solid AE is equal to the solid CF: but the solid AE is equal to the solid SE, as was demonstrated; therefore *the solid* SE *is equal to the solid* CF.

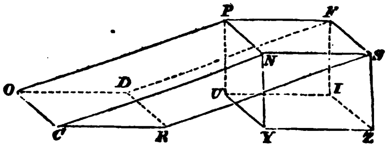
But if the insisting straight lines AG, HK, BE, LM; CN, RS, DF, OP be not at right angles to the bases AB, CD; in this case





likewise, the solid AE shall be equal to the solid CF. From the points G, K, E, M; N, S, F, P, draw the straight lines GQ, KT, EV, MX; NY, SZ, FI, PU, perpendicular to the planes in which are the bases AB, CD (*i*); and let them meet them in the points Q, T, V, X; Y, Z, I, U; and join QT, TV, VX, XQ; YZ, ZI, IU, UY. Then, because GQ, KT are at right angles to the same plane, they are parallel to one another (*k*): and MG, EK are parallels; therefore the planes MQ, ET (of which one passes through MG, GQ, and the other through EK, KT, which are parallel to MG, GQ, and not in the same plane with them) are parallel to one another (*l*): for the same reason, the planes MV, GT are parallel to one another: therefore the solid QE is a parallelopiped. In like manner it may be proved, that the solid YF is a parallelopiped.



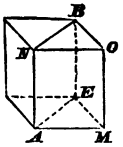


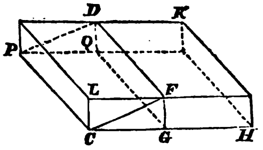
(*k*) XI. 6. (*l*) XI. 15. (*m*) XI. 29 or 30.

But, from what has been demonstrated, the solid EQ is equal to the solid FY, because they are upon equal bases MK, PS, and of the same altitude, and have their insisting straight lines at right angles to the bases: and the solid EQ is equal to the solid AE (*m*), and the solid FY to the solid CF, because they are upon the same bases and of the same altitude; therefore *the solid* AE *is equal to the solid* CF.

PROPOSITION XXXII.

THEOREM.—*If solid parallelopipeds* (AB, CD) *have the same altitude*, they are to one another as their bases.





(*a*) I. 45 cor.
(*b*) XI. 31.
(*c*) XI. 25.
(*d*) XI. 28.

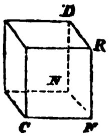
DEMONSTRATION. *To the straight line* FG, *apply the parallelogram* FH, *equal to* AE (*a*), *so that the angle* FGH *may be equal to the angle* LCG; *and upon the base* FH, *complete the solid parallelopiped* GK, one of whose insisting lines is FD, whereby the solids CD, GK must be of the same altitude: therefore the solid AB is equal to the solid GK (*b*), because they are upon equal bases AE, FH, and are of the same altitude: and because the solid parallelopiped CK is cut by the plane DG, which is parallel to its opposite planes, the base HF is to the base FC, as the solid HD to the solid DC (*c*): but the base HF is equal to the base AE, and the solid GK to the solid AB; therefore *as the base* AE *to the base* CF, *so is the solid* AB *to the solid* CD.

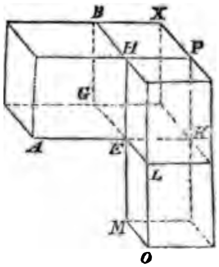
COROLLARY. From this it is manifest, **that prisms upon triangular bases, of the same altitude, are to one another as their bases.**

Let the prisms, the bases of which are the triangles AEM, CFG, and NBO, PDQ the triangles opposite to them, have the same altitude: they shall be to one another as their bases. Complete the parallelograms AE, CF, and the solid parallelopipeds AB, CD, in the first of which let MO, and in the other let GQ be one of the insisting lines. And because the solid parallelopipeds AB, CD have the same altitude, they are to one another as the base AE is to the base CF: wherefore the prisms, which are their (*d*) halves, are to one another, as the base AE to the base CF; that is, as the triangle AEM to the triangle CFG.

PROPOSITION XXXIII.

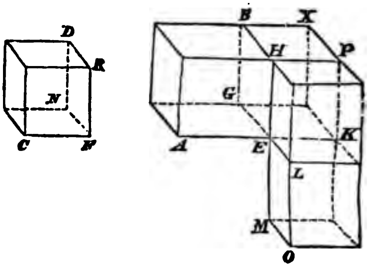
THEOREM.—*If solid parallelopipeds* (AB, CD) *are similar,* **they are one to another in the triplicate ratio of their homologous sides (AE, CF).**





(*a*) XI. 24. (*b*) XI. c.

DEMONSTRATION. *Produce* AE, GE, HE; *and in these produced, take* EK *equal to* CF, EL *equal to* FN, *and* EM *equal to* FR; *and complete the parallelogram* KL, *and the solid* KO. Because KE, EL are equal to CF, FN, each to each, and the angle KEL equal to the angle CFN, because it is equal to the angle AEG, which is equal to CFN, by reason that the solids AB, CD are similar; therefore the parallelogram KL is similar and equal to the parallelogram CN: for the same reason, the parallelogram MK is similar and equal to CR, and also OE to FD. Therefore three parallelograms of the solid KO are equal and similar to three parallelograms of the solid CD: and the three opposite ones in each solid are equal and similar to these (*a*): therefore the solid KO is equal and similar to the solid CD (*b*). *Complete the parallelogram* GK; and upon the bases GK, KL, *complete the solids* EX, LP, so that EH be an insisting straight line in each of them, whereby they must be of the same altitude with the solid AB. And because the solids AB, CD,



(c) I. 6.
(d) XI. 26.
(e) V. Def. 11.

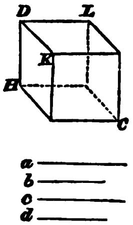
are similar, and by permutation, as AE is to CF, so is EG to FN, and so is EH to FR: but FC is equal to EK, and FN to EL, and FR to EM; therefore, as AE is to EK, so is EG to EL, and so is HE to EM: but as AE is to EK, so is the parallelogram AG to the parallelogram GK (*c*); and as GE is to EL, so is GK to KL (*c*); and as HE is to EM, so is PE to KM (*c*): therefore as the parallelogram AG to the parallelogram GK, so is GK to KL, and PE to KM: but as AG is to GK, so is the solid AB to the solid EX (*d*); and as GK is to KL, so is the solid EX to the solid PL (*d*); and as PE is to KM, so is the solid PL to the solid KO (*d*): and therefore as the solid AB to the solid EX, so is EX to PL, and PL to KO: but if four magnitudes be continual proportionals, the first is said to have to the fourth, the triplicate ratio of that which it has to the second (*e*); therefore the solid AB has to the solid KO, the triplicate ratio of that which AB has to EX: but as AB is to EX, so is the parallelogram AG to the parallelogram GK, and the straight line AE to the straight line EK; wherefore the solid AB has to the solid KO, the triplicate ratio of that which AE has to EK: but the solid KO is equal to the solid CD, and the straight line EK is equal to the straight line CF; therefore *the solid* AB *has to the solid* CD, *the triplicate ratio of that which the side* AE *has to the homologous side* CF.

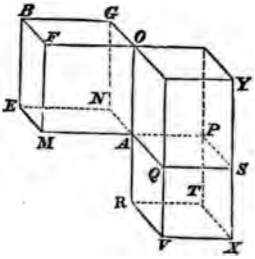
COROLLARY. From this it is manifest, that, if four straight lines be continual proportionals, as the first is to the fourth, so is the solid parallelopiped described from the first to the similar solid similarly described from the second; because the first straight line has to the fourth, the triplicate ratio of that which it has to the second.

PROPOSITION D.

THEOREM.—*If solid parallelopipeds are contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each,* they have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

Demonstration. Let AB, CD be solid parallelopipeds, of which AB is contained by the parallelograms AE, AF, AG, which are equiangular, each to each, to the parallelograms CH, CK, CL, which contain the solid CD: the ratio which the solid AB has to the solid CD, shall be the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH.



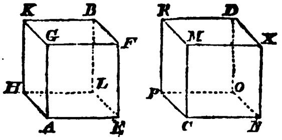


(a) XI. c.
(b) VI. 12.
(c) XI. 32.
(d) XI. 25.
(e) V. Def. A.

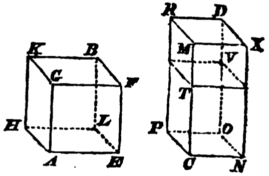
Produce MA, NA, OA *to* P, Q, R, *so that* AP *be equal to* DL, AQ *to* DK, *and* AR *to* DH; *and complete the solid parallelopiped* AX *contained by the parallelograms* AS, AT, AV, similar and equal to CH, CK, CL, each to each: therefore the solid AX is equal to the solid CD (*a*). *Complete likewise the solid* AY, the base of which is AS, and AO one of its insisting straight lines. *Take any straight line* a, *and as* MA *is to* AP, *so make* a *to* b (*b*); *and as* NA *is to* AQ, *so make* b *to* c; *and as* AO *is to* AR, *so make* c *to* d. Then, because the parallelogram AE is equiangular to AS, AE is to AS, as the straight line a is to c, as is demonstrated in the 23rd Prop. Book VI.: and the solids AB, AY, being betwixt the parallel planes BOY, EAS, are of the same altitude; therefore the solid AB is to the solid AY, as the base AE to the base AS (*c*); that is, as the straight line a is to c. And the solid AY is to the solid AX, as the base OQ is to the base QR (*d*); that is, as the straight line OA to AR; that is, as the straight line c to the straight line d. And because the solid AB is to the solid AY, as a is to c, and the solid AY to the solid AX, as c is to d; *ex æquali*, the solid AB is to the solid AX, or CD which is equal to it, as the straight line a is to d. But the ratio of a to d is said to be compounded of the ratios of a to b, b to c, and c to d (*e*), which are the same with the ratios of the sides MA to AP, NA to AQ, and OA to AR, each to each: and the sides AP, AQ, AR are equal to the sides DL, DK, DH, each to each; therefore *the solid* AB *has to the solid* CD, *the ratio which is the same with that which is compounded of the ratios of the sides* AM *to* DL, AN *to* DK, *and* AO *to* DH.

PROPOSITION XXXIV.

THEOREM [1.]—*If solid parallelopipeds are equal,* their bases and altitudes are reciprocally proportional; [2.] *and if the bases and altitudes be reciprocally proportional,* the solid parallelopipeds are equal.



DEMONSTRATION [1.] Let AB, CD be two solid parallelopipeds: and first, let the insisting straight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the bases. If the solid AB be equal to the solid CD, their bases shall be reciprocally proportional to their altitudes; that is, as the base EH is to the base NP, so shall CM be to AG. If the base EH be equal to the base NP, then because the solid AB is likewise equal to the solid CD, CM shall be equal to AG: because if the bases EH, NP be equal, but the altitudes AG, CM be not equal, neither shall the solid AB be equal to the solid CD: but the solids are equal, by the hypothesis; therefore the altitude CM is not unequal to the altitude AG; that is, they are equal. Wherefore, *as the base* EH *to the base* NP, *so is* CM *to* AG.

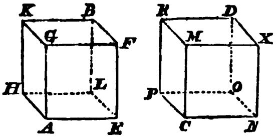


Next, let the bases EH, NP not be equal, but EH greater than the other; then, since the solid AB is equal to the solid CD, CM is therefore greater than AG: for if it be not, neither also in this case would the solids AB, CD be equal, which, by the hypothesis, are equal. *Make then* CT *equal to* AG, *and complete the solid parallelopiped* CV, *of which the base is* NP, *and altitude* CT. Because the solid AB is equal to the solid CD, therefore the solid AB is to the solid CV, as the solid CD to the solid CV (*a*): but as the solid AB to the solid CV, so is the base EH to the base NP (*b*); for the solids AB, CV are of the same altitude: and as the solid CD to CV, so is the base MP to the base PT (*c*), and so is the straight line MC to CT (*d*): and CT is equal to AG; therefore as the base EH to the base NP, so is MC to AG

(*a*) V. 7.
(*b*) XI. 32.
(*c*) XI. 25.
(*d*) V. 1.
(*e*) V. A.
(*f*) XI. 31.
(*g*) V. 9.

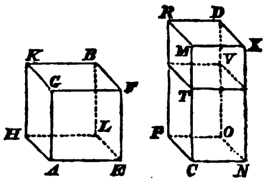
Wherefore *the bases of the solid parallelopipeds* AB, CD *are reciprocally proportional to their altitudes.*

[2.] Let now the bases of the solid parallelopipeds AB, CD be reciprocally proportional to their altitudes, viz. as the base EH is to the base NP, so let CM be to AG: the solid AB shall be equal to the solid CD.



If the base EH be equal to the base NP, then, since EH is to NP as the altitude of the solid CD is to the altitude of the solid AB, therefore the altitude of CD is equal to the altitude of AB (*e*): but solid parallelopipeds upon equal bases, and of the same altitude, are equal to one another (*f*); therefore *the solid* AB *is equal to the solid* CD.

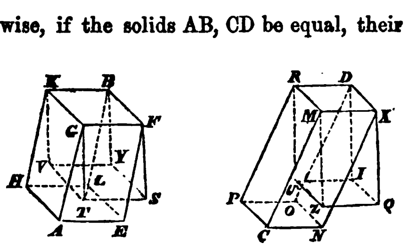
But let the bases EH, NP be unequal, and let EH be the greater of the two: therefore, since, as the base EH to the base NP, so is CM the altitude of the solid CD to AG the altitude of AB, CM is greater than AG (*e*). Therefore, as before, *take* CT *equal to* AG, and *complete the solid* CV. And because the base EH is to the base NP, as CM to AG, and that AG is equal to CT, therefore the base EH is to the base NP, as MC to CT. But as the base EH is to NP, so is the solid AB to the solid CV (*b*); for the solids AB, CV are of the same altitude: and as MC is to CT, so is the base MP to the base PT (*d*), and the solid CD to the solid CV (*c*): therefore as the solid AB is to the solid CV, so is the solid CD to the solid CV; that is, each of the solids AB, CD has the same ratio to the solid CV; and therefore *the solid* AB *is equal to the solid* CD (*g*).



Second general case. Let the insisting straight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the bases of the solids.

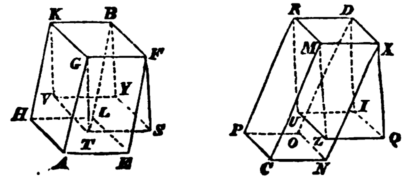
[1.] In this case, likewise, if the solids AB, CD be equal, their bases shall be reciprocally proportional to their altitudes, viz. the base EH shall be to the base NP, as the altitude of the solid CD is to the altitude of the solid AB.

From the points F, B, K, G; X, D, R, M,



draw perpendiculars to the planes in which are the bases EH, NP, *meeting those planes in the points* S, Y, V, T; Q, I, U, Z; *and complete the solids* FV, XU, which are parallelopipeds, as was proved in the last part of Prop. 31, of this book.

Because the solid AB is equal to the solid CD, and that the solid AB is equal to the solid BT (*h*), for they are upon the same base FK, and of the same altitude; and that the solid CD is equal to the solid DZ (*h*), being upon the same base XR, and of the same altitude; therefore the solid BT is equal to the solid DZ: but the bases are reciprocally proportional to the altitudes of equal solid parallelopipeds of which the insisting straight lines are at right angles to their bases, as before was proved; therefore as the base FK to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT: and the base FK is equal to the base EH, and the base XR to the base NP; wherefore, as the base EH is to the base NP, so is the altitude of the solid DZ to the altitude of the solid BT; but the altitudes of the solids DZ, DC, as also of the solids BT, BA, are the same; therefore as the base EH to the base NP, so is the altitude of the solid CD to the altitude of the solid AB; that is, *the bases of the solid parallelopipeds* AB, CD *are reciprocally proportional to their altitudes.*

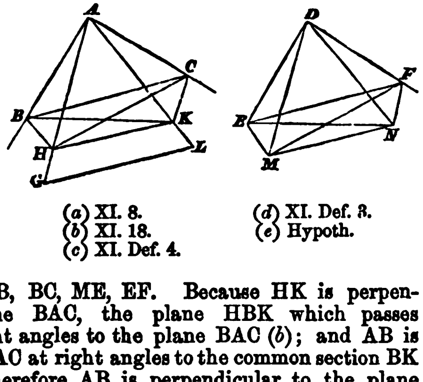


(*h*) XI. 29 or 30.

[2.] Next, let the bases of the solids AB, CD be reciprocally proportional to their altitudes, viz. the base EH is to the base NP, as the altitude of the solid CD is to the altitude of the solid AB: the solid AB shall be equal to the solid CD. The same construction being made; because, as the base EH is to the base NP, so is the altitude of the solid CD to the altitude of the solid AB; and that the base EH is equal to the base FK, and NP to XR; therefore the base FK is to the base XR, as the altitude of the solid CD to the altitude of AB; but the altitudes of the solids AB, BT are the same, as also of CD and DZ; therefore as the base FK is to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT: wherefore the bases of the solids BT, DZ are reciprocally proportional to their altitudes: and their insisting straight lines are at right angles to the bases; wherefore, as was before proved, the solid BT is equal to the solid DZ: but BT is equal to the solid BA (*h*), and DZ to the solid DC, because they are upon the same bases, and of the same altitude; therefore *the solid* AB *is equal to the solid* CD.

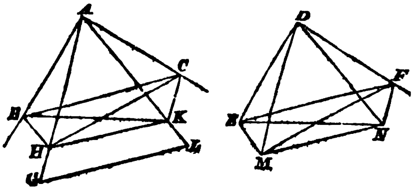
PROPOSITION XXXV.

THEOREM.—*If, from the vertices* (A and D) *of two equal plane angles* (BAC, EDF), *there be drawn two straight lines* (AG, DM) *elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each* (GAB to MDE, and GAC to MDF); *and if in the lines* (AG, DM) *above the planes there be taken any points* (G, M), *and from them perpendiculars* (GL, MN) *be drawn to the planes in which are the first-named angles* (BAC, EDF); *and from the points* (L, N) *in which they meet the planes, straight lines* (LA, ND) *be drawn to the vertices of the angles first-named;* these straight lines shall contain equal angles (GAL, MDN) with the straight lines which are above the planes of the angles.



(*a*) XI. 8.
(*b*) XI. 18.
(*c*) XI. Def. 4.
(*d*) XI. Def. 3.
(*e*) Hypoth.

DEMONSTRATION. *Make* AH *equal to* DM, *and through* H *draw* HK *parallel to* GL: but GL is perpendicular to the plane BAC; wherefore HK is perpendicular to the same plane (*a*). *From the points* K, N, *to the straight lines* AB, AC, DE, DF, *draw perpendiculars* KB, KC, NE, NF, *and join* HB, BC, ME, EF. Because HK is perpendicular to the plane BAC, the plane HBK which passes through HK is at right angles to the plane BAC (*b*); and AB is drawn in the plane BAC at right angles to the common section BK of the two planes; therefore AB is perpendicular to the plane HBK (*c*), and makes right angles with every straight line meeting it in that plane (*d*): but BH meets it in that plane; therefore ABH is a right angle: for the same reason, DEM is a right angle, and is therefore equal to the angle ABH: and the angle HAB is equal to the angle MDE (*e*); therefore in the two triangles HAB, MDE, there are two angles in one, equal to two angles in the other, each to each, and one side equal to one side, opposite to one of the equal angles in each, viz. HA equal to DM; therefore the remaining sides are equal, each to each (*f*), wherefore AB is equal to DE. In the same manner, if HC and MF be joined, it may be demonstrated, that AC is equal to DF: therefore, since AB is equal to DE, BA and AC are equal to ED and DF, each to each; and the angle BAC is equal to the angle EDF (*e*); wherefore the base BC is equal to the base EF (*g*), and the remaining angles to the remaining angles; therefore the angle ABC is equal to the angle DEF: and the right angle ABK is equal to the right angle DEN; whence the remaining angle CBK is equal to the remaining angle FEN: for the same reason, the angle BCK is equal to the angle EFN; therefore in the two triangles BCK, EFN there are two angles in one, equal to two angles in the other, each to each, and one side equal to one side adjacent to the equal angles in each, viz. BC equal to EF; therefore the other sides are equal to the other sides; BK then is equal to EN: but AB is equal to DE; wherefore AB, BK are equal to DE, EN, each to each; and they contain right angles; wherefore the base AK is equal to the base DN. And since AH is equal to DM, the square on AH is equal to the square on DM: but the squares on AK, KH are equal to the square on AH, because AKH is a right angle (*h*); and the squares on DN, NM are equal to the square on DM, for DNM is a right angle: wherefore the squares on AK, KH are equal to the squares on DN, NM: and of these the square on AK is equal to the square on DN; therefore the remaining square on KH is equal to the remaining square on NM; and the straight line KH to the straight line NM; and because HA, AK are equal to MD, DN, each to each, and the base HK to the base MN, as has been proved, therefore *the angle* HAK, *that is*, GAL, *is equal to the angle* MDN (*i*).

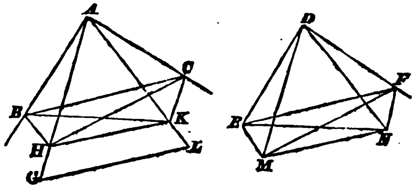


(*e*) Hypoth.
(*f*) I. 26.
(*g*) I. 4.
(*h*) I. 47.
(*i*) I. 8.

Corollary. From this it is manifest, that if from the vertices of two equal plane angles, there be elevated two equal straight lines containing equal angles with the sides of the angles, each to each; the perpendiculars drawn from the extremities of the equal straight lines to the planes of the first angles, are equal to one another.

Scholium. Of this Corollary another demonstration may be given, as follows:—

Let the plane angles BAC, EDF be equal to one another; and let AH, DM be two equal straight lines above the planes of the angles, containing equal angles with BA, AC; ED, DF, each to each, viz. the angle HAB equal to MDE, and HAC equal to the angle MDF; and from H, M let HK, MN be perpendiculars to the planes BAC, EDF: HK shall be equal to MN.

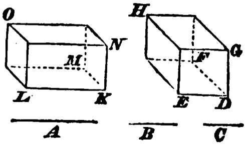


(*a*) XI. 18.

Because the solid angle at A is contained by the three plane angles BAC, BAH, HAC, which are, each to each, equal to the three plane angles EDF, EDM, MDF, containing the solid angle at D; the solid angles at A and D are equal, and therefore coincide with one another; to wit, if the plane angle BAC be applied to the plane angle EDF, the straight line AH coincides with DM, as was shown in Prop. B, of this book: and because AH is equal to DM, the point H coincides with the point M: wherefore HK, which is perpendicular to the plane BAC, coincides with MN, which is perpendicular to the plane EDF, because these planes coincide with one another (*a*). Therefore HK *is equal to* MN.

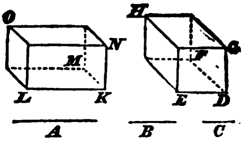
PROPOSITION XXXVI.

Theorem.—*If three straight lines* (A, B, C) *be proportionals*, the solid parallelopiped described from all three, as its sides, is equal to the equilateral parallelopiped described from the mean proportional (B), one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.



(*a*) XI. 26.

Demonstration. *Take a solid angle* D, *contained by three plane angles* EDF, FDG, GDE; *and make each of the straight lines* ED, DF, DG *equal to* B, *and complete the solid parallelopiped* DH: *make* LK *equal to* A, *and at the point* K, *in the straight line* LK, *make a solid angle contained by the three plane angles* LKM, MKN, NKL, *equal to the angles* EDF, FDG, GDE, *each to each* (*a*); *and make* KN *equal to* B, *and* KM *equal to* C; *and complete the solid parallelopiped* KO. And because, as A is to B, so is B to C, and that A is

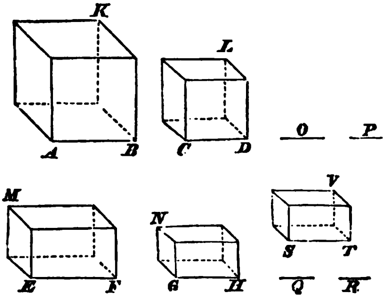


equal to LK, and B is equal to each of the straight lines DE, DF, and C is equal to KM; therefore LK is to ED, as DF to KM; that is, the sides about the equal angles are reciprocally proportional; therefore the parallelogram LM is equal to EF (*b*): and because EDF, LKM are two equal plane angles, and the two equal straight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their sides; therefore the perpendiculars from the points G, N, to the planes EDF, LKM are equal to one another (*c*): therefore the solids KO, DH are of the same altitude; and they are upon equal bases LM, EF; and therefore *they are equal to one another* (*d*): *but the solid* KO *is described from the three straight lines* A, B, C, *and the solid* DH *from the straight line* B.

(*b*) VI. 14.
(*c*) XI. 35, cor.
(*d*) XI. 31.

PROPOSITION XXXVII.

Theorem [1.]—*If four straight lines* (AB, CD, EF, GH) *be proportionals*, the similar solid parallelopipeds (AK, CL, FM, HN) similarly described from them shall also be proportionals: *and if the similar parallelopipeds similarly described from four straight lines be proportionals*, the straight lines shall be proportionals.



Demonstration [1.] Make AB, CD, O, P, continual proportionals, as also EF, GH, Q, R (*a*): and because as AB is to CD, so is EF to GH; and that CD is to O, as GH to Q (*b*), and O is to P, as Q to R; therefore, *ex æquali*, AB is to P, as EF to R (*c*): but as AB is to P, so is the solid AK to the solid CL (*d*); and as EF is to R, so is the solid FM to the solid HN (*d*):

(*a*) VI. 11.
(*b*) V. 11.
(*c*) V. 22.
(*d*) XI. 33, cor.
(*e*) XI. 27.
(*f*) V. 9.

therefore *as the solid* AK *is to the solid* CL, *so is the solid* FM *to the solid* HN (*b*).

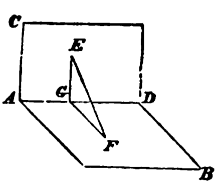
[2.] Next, let the solid AK be to the solid CL, as the solid FM is to the solid HN: the straight line AB shall be to CD, as EF is to GH.

Take as AB *is to* CD, *so is* EF *to* ST, *and from* ST *describe a solid parallelopiped* SV *similar and similarly situated to either of the solids* FM, HN (*e*). And because AB is to CD, as EF is to ST, and that from AB, CD the solid parallelopipeds AK, CL are similarly described; and in like manner the solids FM, SV from the straight lines EF, ST; therefore AK is to CL, as FM is to SV; but, by the hypothesis, AK is to CL, as FM to HN; therefore HN is equal to SV (*f*): but it is likewise similar and similarly situated to SV; therefore the planes which contain the solids HN, SV are similar and equal, and their homologous sides GH, ST equal to one another: and because as AB is to CD, so is EF to ST, and that ST is equal to GH, therefore AB *is to* CD, *as* EF *is to* GH.

PROPOSITION XXXVIII.

THEOREM.—"*If a plane* (CD) *be perpendicular to another plane* (AB), *and a straight line be drawn from a point* (E) *in one of the planes* (CD) *perpendicular to the other plane* (AB), this straight line shall fall on the common section (AD) of the planes."

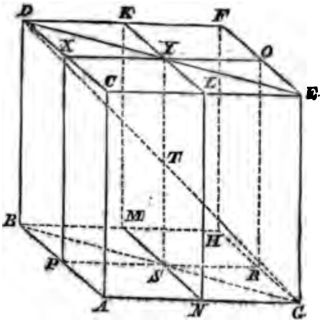
DEMONSTRATION. "For if it does not, let it, if possible, fall elsewhere, as EF; and let it meet the plane AB in the point F; *and from* F *draw, in the plane* AB, *a perpendicular* FG *to* DA (*a*), *which is also perpendicular to the plane* CD (*b*); *and join* EG. Then, because FG is perpendicular to the plane CD, and the straight line EG which is in that plane, meets it, therefore FGE is a right angle (*c*): but EF is also at right angles to the plane AB, and therefore EFG is a right angle: wherefore two of the angles of the triangle EFG are equal together to two right angles; which is absurd (*d*); therefore the perpendicular from the point E to the plane AB, *does not fall elsewhere than upon the straight line* AD; *that is, it therefore falls upon it.*"



(*a*) I. 12.
(*b*) XI. Def. 4.
(*c*) XI. Def. 3.
(*d*) I. 17.

PROPOSITION XXXIX.

Theorem.—*In a solid parallelopiped, if the sides of two of the opposite planes be divided, each into two equal parts,* the common section of the planes passing through the points of division, and the diameter of the solid parallelopiped, cut each other into two equal parts.



(*a*) I. 33.
(*b*) XI. 9.
(*c*) I. 29.
(*d*) I. 4.
(*e*) I. 14.
(*f*) I. 15.
(*g*) I. 26.

Demonstration. Let the sides of the opposite planes CF, AH of the solid parallelopiped AF, be divided each into two equal parts in the points K, L, M, N; X, O, P, R; *and join* KL, MN, XO, PR: and because DK, CL are equal and parallel, KL is parallel to DC (*a*): for the same reason, MN is parallel to BA: and BA is parallel to DC; therefore, because KL, BA are each of them parallel to DC, and not in the same plane with it, KL is parallel to BA (*b*): and because KL, MN are each of them parallel to BA, and not in the same plane with it, KL is parallel to MN (*b*): wherefore KL, MN are in one plane. In like manner it may be proved, that XO, PR are in one plane. Let YS be the common section of the planes KN, XR; and DG the diameter of the solid parallelopiped AF: YS and DG shall meet, and cut one another into two equal parts.

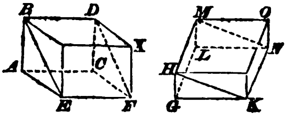
Join DY, YE, BS, SG. Because DX is parallel to OE, the alternate angles DXY, YOE are equal to one another (*c*): and because DX is equal to OE, and XY to YO, and that they contain equal angles, the base DY is equal to the base YE (*d*), and the other angles are equal; therefore the angle XYD is equal to the angle OYE, and DYE is a straight line (*e*): for the same reason, BSG is a straight line, and BS equal to SG. And because CA is equal and parallel to DB, and also equal and parallel to EG, therefore DB is equal and parallel to EG (*b*): and DE, BG join their extremities; therefore DE is equal and parallel to BG (*a*): and DG, YS are drawn from points in the one, to points in the other, and are therefore in one plane: whence it is manifest, that DG, YS must meet one another: let them meet in T. And because DE is parallel to BG, the alternate angles EDT, BGT are equal (*c*): and the angle DTY is equal to the angle GTS (*f*): therefore in the triangles DTY, GTS, there are two angles in the one, equal to two angles in the other, and one side equal to one side, opposite to two of the equal angles, viz. DY to GS, for they are the halves of DE, BG; therefore the remaining sides are equal, each to each (*g*): wherefore DT *is equal to* TG, *and* YT *equal to* TS.

PROPOSITION XL.

THEOREM.—*If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram, and the base of the other a triangle: if the parallelogram be double of the triangle*, the prisms shall be equal to one another.

DEMONSTRATION. Let the prisms ABCDEF, GHKLMN be of the same altitude, the first whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN, and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK, for its base: if the parallelogram AF be double of the triangle GHK, the prism ABCDEF shall be equal to the prism GHKLMN.

Complete the solids AX, GO: and because the parallelogram AF is double of the triangle GHK, and the parallelogram HK double of the same triangle, therefore the parallelogram AF is equal to HK (*a*): but solid parallelopipeds upon equal bases, and of the same altitude, are equal to one another (*b*); therefore the solid AX is equal to the solid GO: and the prism ABCDEF is half of the solid AX (*c*): and the prism GHKLMN half of the solid GO (*c*): therefore *the prism* ABCDEF *is equal to the prism* GHKLMN.



(*a*) I. 34.
(*b*) XI. 31.
(*c*) XI. 28.

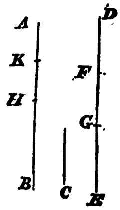
THE
ELEMENTS OF EUCLID.

BOOK XII.

LEMMA I.

Theorem.—*If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half, and so on;* there shall at length remain a magnitude less than the least of the proposed magnitudes.

Demonstration. Let AB and C be two unequal magnitudes, of which AB is the greater: if from AB there be taken more than its half, and from the remainder more than its half, and so on; there shall at length remain a magnitude less than C.



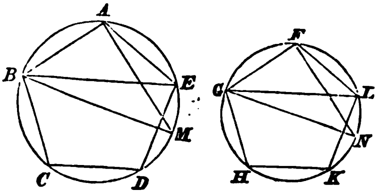
For C may be multiplied so as at length to become greater than AB. Let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB, take BH greater than its half; and from the remainder AH, take HK greater than its half, and so on, until there be as many divisions in AB as there are in DE. And because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half, therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK. And FD is equal to C, therefore C is greater than AK; that is, AK *is less than* C.

Corollary. And if only the halves be taken away, the same thing may, in the same way, be demonstrated.

Scholium. This is the first proposition in the 10th book, and being necessary to some of the propositions of this book, it is here inserted.

PROPOSITION I.

Theorem.—*Similar polygons* (ABCDE, FGHKL) *inscribed in circles*, are to one another as the squares on their diameters.



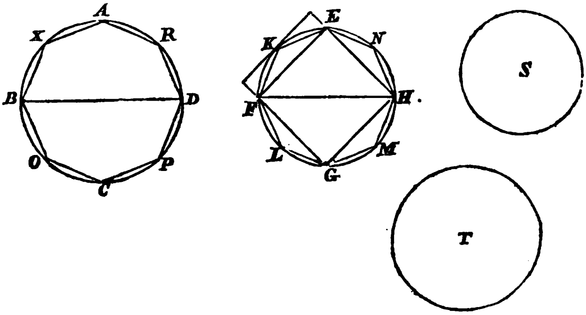
(a) VI. Def. 1.
(b) III. 21.
(c) III. 31.
(d) VI. 4.
(e) V. Def. 10 and 22.
(f) VI. 20.

Demonstration. Let BM, GN be the diameters of the circles; *join* BE, AM; GL, FN: and because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL (*a*), and as BA is to AE, so is GF to FL: therefore the two triangles BAE, GFL having an angle in one, equal to an angle in the other, and the sides about the equal angles proportionals, are equiangular; and therefore the angle AEB is equal to the angle FLG: but AEB is equal to AMB, because they stand upon the same circumference (*b*): and the angle FLG is, for the same reason, equal to the angle FNG: therefore also the angle AMB is equal to FNG; and the right angle BAM is equal to the right angle GFN (*c*); wherefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another; therefore as BM is to GN, so is BA to GF (*d*); and therefore the duplicate ratio of BM to GN, is the same with the duplicate ratio of BA to GF (*e*): but the ratio of the square on BM to the square on GN, is the duplicate ratio of that which BM has to GN; and the ratio of the polygon ABCDE to the polygon FGHKL is the duplicate of that which BA has to GF (*f*): therefore *as the square on* BM *is to the square on* GN, *so is the polygon* ABCDE *to the polygon* FGHKL.

PROPOSITION II.

THEOREM.—*Circles* (AC, EG) are to one another as the squares on their diameters.

DEMONSTRATION. For if it be not so, the square on BD must be to the square on FH, as the circle AC is to some space either less than the circle EG, or greater than it. First let it be to a space S less than the circle EG; *and in the circle* EG *describe the square*



(a) IV. 6.
(b) I. 41.

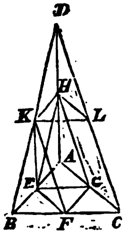
(c) XII. 1.
(d) Hypoth.

(e) V. 11.
(f) V. 14.

EFGH (a). This square is greater than half of the circle EG; because, if, through the points E, F, G, H there be drawn tangents to the circle, the square EFGH is half of the square described about the circle (b): and the circle is less than the square described about it, therefore the square EFGH is greater than half of the circle. *Divide the circumferences* EF, FG, GH, HE *each into two equal parts in the points* K, L, M, N, *and join* EK, KF, FL, LG, GM, MH, HN, NE: therefore each of the triangles EKF, FLG, GMH, HNE, is greater than half of the segment of the circle in which it stands; because, if straight lines touching the circle be drawn through the points K, L, M, N, and the parallelograms upon the straight lines EF, FG, GH, HE be completed, each of the triangles EKF, FLG, GMH, HNE is the half of the parallelogram in which it is (b); but every segment is less than the parallelogram in which it is: wherefore each of the triangles EKF, FLG, GMH, HNE is greater than half the segment of the circle which contains it. Again, if the remaining circumferences be divided each into two equal parts, and their extremities be joined by straight lines, by continuing to do this, there will at length remain segments of the circle, which together are less than the excess of the circle EG above the space S; because, by the preceding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the least of the proposed magnitudes. Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those that remain, and are together less than the excess of the circle EG above S: therefore the rest of the circle, viz. the polygon EKFLGMHN, is greater than the space S. *Describe likewise in the circle* AC *the polygon* AXBOCPDR *similar to the polygon* EKFLGMHN: as therefore the square on BD is to the square on FH, so is the polygon AXBOCPDR to the polygon EKFLGMHN (*c*): but the square on BD is also to the square on FH, as the circle AC is to the space S (*d*); therefore as the circle AC is to the space S, so is the polygon AXBOCPDR to the polygon EKFLGMHN (*e*): but the circle AC is greater than the polygon contained in it; wherefore the space S is greater than the polygon EKFLGMHN (*f*): but it is likewise less, as has been demonstrated; which is impossible: therefore the square on BD is not to the square on FH, as the circle AC is to any space less than the circle EG. In the same manner it may be demonstrated, that neither is the square on FH to the square on BD, as the circle EG is to any space less than the circle AC. Nor is the square on BD to the square on FH, as the circle AC is to any space greater than the circle EG. For if possible, let it be so to T, a space greater than the circle EG: therefore inversely, as the square on FH is to the square on BD, so is the space T to the circle AC: but as the space T is to the circle AC, so is the circle EG to some space, which must be less than the circle AC (*f*), because the space T is greater, by hypothesis, than the circle EG; therefore as the square on FH is to the square on BD, so is the circle EG to a space less than the circle AC, which has been demonstrated to be impossible: therefore the square on BD is not to the square on FH, as the circle AC is to any space greater than the circle EG: and it has been demonstrated that neither is the square on BD to the square on FH, as the circle AC to any space less than the circle EG: wherefore, *as the square on* BD *is to the square on* FH, *so is the circle* AC *to the circle* EG.

PROPOSITION III.

THEOREM.—*Every pyramid having a triangular base* (ABC) may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than half of the whole pyramid.



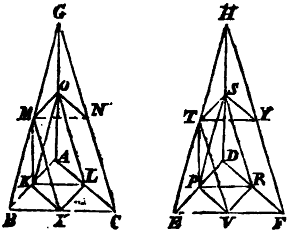
(a) VI. 2.
(b) I. 34.
(c) I. 29.
(d) I. 4.
(e) XI. 10.
(f) XI. C.
(g) VI. 4.
(h) VI. 21.
(i) XI. B. and Def. 11.
(k) I. 41.
(l) XI. 40.
(m) XI. 15.

DEMONSTRATION. *Divide* AB, BC, CA, AD, DB, DC *each into two equal parts in the points* E, F, G, H, K, L, *and join* EH, EG, GH, HK, KL, LH, EK, KF, FG. Because AE is equal to EB, and AH to HD, HE is parallel to DB (*a*): for the same reason, HK is parallel to AB; therefore HEBK is a parallelogram, and HK equal to EB (*b*): but EB is equal to AE; therefore also AE is equal to HK: and AH is equal to HD; wherefore EA, AH are equal to KH, HD, each to each; and the angle EAH is equal to the angle KHD (*c*); therefore the base EH is equal to the base KD, and the triangle AEH equal and similar to the triangle HKD (*d*): for the same reason, the triangle AGH is equal and similar to the triangle HLD. Again, because the two straight lines EH, HG, which meet one another, are parallel to KD, DL, that meet one another and are not in the same plane with them, they contain equal angles (*e*); therefore the angle EHG is equal to the angle KDL: and because EH, HG are equal to KD, DL, each to each, and the angle EHG equal to the angle KDL; therefore the base EG is equal to the base KL; and the triangle EHG equal and similar to the triangle KDL (*d*): for the same reason, the triangle AEG is also equal and similar to the triangle HKL: therefore the pyramid, of which the base is the triangle AEG, and of which the vertex is the point H, is equal and similar to the pyramid, the base of which is the triangle KHL, and vertex the point D (*f*). And because HK is parallel to AB, a side of the triangle ADB, the triangle ADB is equiangular to the triangle HDK, and their sides are proportionals (*g*); therefore the triangle ADB is similar to the triangle HDK: and for the same reason, the triangle DBC is similar to the triangle DKL; and the triangle ADC to the triangle HDL; and also the triangle ABC to the triangle AEG: but the triangle AEG is similar to the triangle HKL, as before was proved; therefore the triangle ABC is similar to the triangle HKL (*h*): and therefore the pyramid of which the base is the triangle ABC, and vertex the point D, is similar to the pyramid of which the base is the triangle HKL, and vertex the same point D (*i*): but the pyramid of which the base is the triangle HKL, and vertex the point D, is similar, as has been proved, to the pyramid the base of which is the triangle AEG, and vertex the point H; wherefore the pyramid, the base of which is the triangle ABC, and vertex the point D, is similar to the pyramid of which the base is the triangle AEG and vertex H: therefore each of the pyramids AEGH, HKLD is similar to the whole pyramid ABCD. And because BF is equal to FC, the parallelogram EBFG is double of the triangle GFC (*k*): but when there are two prisms of the same altitude, of which one has a parallelogram for its base, and the other a triangle that is half of the parallelogram, these prisms are equal to one another (*l*); therefore the prism having the parallelogram EBFG for its base, and the straight line KH opposite to it, is equal to the prism having the triangle GFC for its base, and the triangle HKL opposite to it; for they are of the same altitude, because they are between the parallel planes ABC, HKL (*m*): and it is manifest that each of these prisms is greater than either of the pyramids of which the triangles AEG, HKL are the bases, and the vertices the points H, D; because, if EF be joined, the prism having the parallelogram EBFG for its base, and KH the straight line opposite to it, is greater than the pyramid of which the base is the triangle EBF, and vertex the point K: but this pyramid is equal to the pyramid, the base of which is the triangle AEG, and vertex the point H (*f*); because they are contained by equal and similar planes: wherefore the prism having the parallelogram EBFG for its base, and opposite side KH, is greater than the pyramid of which the base is the triangle AEG, and vertex the point H: and the prism of which the base is the parallelogram EBFG, and opposite side KH, is equal to the prism having the triangle GFC for its base, and HKL the triangle opposite to it; and the pyramid of which the base is the triangle AEG, and vertex H, is equal to the pyramid of which the base is the triangle HKL, and vertex D: therefore the two prisms before mentioned are greater than the two pyramids of which the bases are the triangles AEG, HKL, and vertices the points H, D. Therefore, *the whole pyramid of which the base is the triangle* ABC, *and vertex the point* D, *is divided into two equal pyramids similar to one another, and to the whole pyramid; and into two equal prisms; and the two prisms are together greater than half of the whole pyramid.*

PROPOSITION IV.

THEOREM.—*If there be two pyramids* (ABCG, DEFH) *of the same altitude upon triangular bases* (ABC, DEF), *and each of them be divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on:* as the base (ABC) of one of the first two pyramids is to the base (DEF) of the other, so shall all the prisms in one of them (ABCG) be to all the prisms in the other (DEFH), that are produced by the same number of divisions.

DEMONSTRATION. *Make the same construction as in the foregoing proposition:* and because BX is equal to XC, and AL to LC, therefore XL is parallel to AB (*a*), and the triangle ABC similar to the triangle LXC: for the same reason, the triangle DEF is similar to RVF. And because BC is double of CX, and EF double of FV; therefore BC is to CX, as EF is to FV (*b*): and upon BC, CX are described the similar and similarly situated rectilineal figures ABC, LXC: and upon EF, FV, in like manner are described the similar figures DEF, RVF: therefore, as the triangle ABC is to the triangle LXC, so is the triangle DEF to the triangle RVF (*c*), and, by permutation, as the triangle ABC is to the triangle DEF, so is the triangle LXC to the triangle RVF. And because the planes ABC, OMN, as also the planes DEF, STY, are parallel (*d*), the perpendiculars drawn from the points G, H to the bases ABC, DEF, which, by the hypothesis, are equal to one another, shall be cut each into two equal parts by the planes OMN, STY (*e*), because the straight lines GC, HF are cut into two equal parts in the points N, Y, by the same planes: therefore the prisms LXCOMN, RVFSTY are of the same altitude; and therefore, as



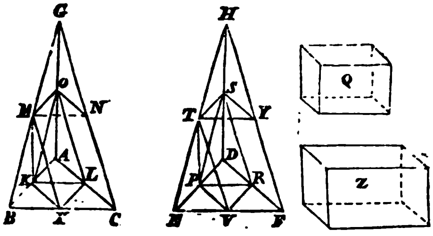
(*a*) VI. 2.
(*b*) V. c.
(*c*) VI. 22.
(*d*) XI. 15.
(*e*) XI. 17.
(*f*) XI. 32, cor.
(*g*) V. 7.

the base LXC is to the base RVF; that is, as the triangle ABC is to the triangle DEF, so is the prism having the triangle LXC for its base, and OMN the triangle opposite to it, to the prism of which the base is the triangle RVF, and the opposite triangle STY (*f*): and because the two prisms in the pyramid ABCG are equal to one another, and also the two prisms in the pyramid DEFH equal to one another; as the prism of which the base is the parallelogram KBXL and opposite side MO, is to the prism having the triangle LXC for its base, and OMN the triangle opposite to it; so is the prism of which the base is the parallelogram PEVR, and opposite side TS, to the prism of which the base is the triangle RVF, and opposite triangle STY (*g*): therefore, *componendo*, as the prisms KBXLMO, LXCOMN together, are to the prism LXCOMN; so are the prisms PEVRTS, RVFSTY to the prism RVFSTY; and, *permutando*, as the prisms KBXLMO, LXCOMN are to the prisms PEVRTS, RVFSTY; so is the prism LXCOMN to the prism RVFSTY: but as the prism LXCOMN to the prism RVFSTY, so is, as has been proved, the base ABC to the base DEF; therefore, as the base ABC is to the base DEF, so are the two prisms in the pyramid ABCG to the two prisms in the pyramid DEFH: and likewise if the pyramids now made, for example, the two OMNG, STYH, be divided in the same manner; as the base OMN is to the base STY, so are the two prisms in the pyramid OMNG to the two prisms in the pyramid STYH: but the base OMN is to the base STY, as the base ABC is to the base DEF; therefore, *as the base* ABC *is to the base* DEF, *so are the two prisms in the pyramid* ABCG *to the two prisms in the pyramid* DEFH; *and so are the two prisms in the pyramid* OMNG *to the two prisms in the pyramid* STYH; *and so are all four to all four: and the same thing may be shown of the prisms made by dividing the pyramids* AKLO *and* DPRS, *and of all made by the same number of divisions.*

PROPOSITION V.

THEOREM.—*Pyramids* (ABCG, DEFH) *of the same altitude which have triangular bases* (ABC, DEF) are to one another as their bases.

DEMONSTRATION. For if it be not so, the base ABC must be to the base DEF as the pyramid ABCG to a solid either less than the pyramid DEFH, or greater than it. First, let it be to a solid less than it, viz. to the solid Q; *and divide the pyramid* DEFH *into two equal pyramids, similar to the whole, and into two equal*



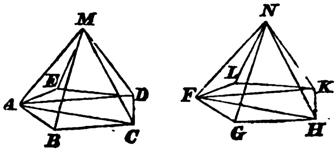
(a) XII. 3.
(b) XII. Lemma 1.
(c) XII. 4.
(d) V. 14.

prisms; therefore these two prisms are greater than the half of the whole pyramid (*a*). And again, *let the pyramids made by this division be in like manner divided*, and so on (*b*), until the pyramids which remain undivided in the pyramid DEFH be all of them together less than the excess of the pyramid DEFH above the solid Q: let these, for example, be the pyramids DPRS, STYH: therefore the prisms, which make the rest of the pyramid DEFH, are greater than the solid Q. *Divide likewise the pyramid* ABCG *in the same manner, and into as many parts as the pyramid* DEFH: therefore as the base ABC to the base DEF, so are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH (*c*): but as the base ABC to the base DEF, so, by hypothesis, is the pyramid ABCG to the solid Q: and therefore, as the pyramid ABCG to the solid Q, so are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH; but the pyramid ABCG is greater than the prisms contained in it; wherefore also the solid Q is greater than the prisms in the pyramid DEFH (*d*): but it is also less, which is impossible: therefore the base ABC is not to the base DEF, as the pyramid ABCG to any solid which is less than the pyramid DEFH. In the same manner it may be demonstrated, that the base DEF is not to the base ABC, as the pyramid DEFH to any solid which is less than the pyramid ABCG. Nor can the base ABC be to the base DEF, as the pyramid ABCG to any solid which is greater than the pyramid DEFH. For if it be possible, let it be so to a greater, viz. the solid Z. And because the base ABC is to the base DEF, as the pyramid ABCG to the solid Z; by inversion, as the base DEF is to the base ABC, so is the solid Z to the pyramid ABCG: but as the solid Z is to the pyramid ABCG, so is the pyramid DEFH to some solid, which must be less than the pyramid ABCG (*d*), because the solid Z is greater than the pyramid DEFH; and therefore, as the base DEF is to the base ABC, so is the pyramid DEFH to a solid less than the pyramid ABCG; the contrary to which has been proved: therefore the base ABC is not to the base DEF, as the pyramid ABCG to any solid which is greater than the pyramid DEFH. And it has been proved, that neither is the base ABC to the base DEF, as the pyramid ABCG to any solid which is less than the pyramid DEFH;

therefore, *as the base* ABC *is to the base* DEF, *so is the pyramid* ABCG *to the pyramid* DEFH.

PROPOSITION VI.

THEOREM.—*Pyramids* (ABCDEM, FGHKLN) *of the same altitude which have polygons* (ABCDE, FGHKL) *for their bases*, are to one another as their bases.

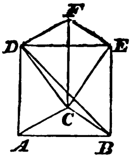


(*a*) XII. 5.
(*b*) V. 24, cor. 2.
(*c*) V. 22.

DEMONSTRATION. *Divide the base* ABCDE *into the triangles* ABC, ACD, ADE: *and the base* FGHKL *into the triangles* FGH, FHK, FKL: *and upon the bases* ABC, ACD, ADE, *let there be as many pyramids of which the common vertex is the point* M, *and upon the remaining bases as many pyramids having their common vertex in the point* N. Therefore, since the triangle ABC is to the triangle FGH, as the pyramid ABCM to the pyramid FGHN (*a*); and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN; and also the triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the first antecedents to their common consequent, so are all the other antecedents to their common consequent (*b*); that is, as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN: and for the same reason, as the base FGHKL to the base FGH, so is the pyramid FGHKLN to the pyramid FGHN; and, by inversion, as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN: then, because, as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN; and as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN; therefore, *ex æquali, as the base* ABCDE *is to the base* FGHKL, *so is the pyramid* ABCDEM *to the pyramid* FGHKLN (*c*).

PROPOSITION VII.

THEOREM.—*Every prism* (ABCDEF) *having a triangular base* (ABC) may be divided into three pyramids that have triangular bases, and are equal to one another.



(a) I. 34.
(b) XII. 5.
(c) XII. 6.

Demonstration. *Join* BD, EC, CD: and because ABED is a parallelogram of which BD is the diagonal, the triangle ABD is equal to the triangle EBD (*a*); therefore the pyramid of which the base is the triangle ABD, and vertex the point C, is equal to the pyramid of which the base is the triangle EBD, and vertex the point C (*b*): but this pyramid is the same with the pyramid the base of which is the triangle EBC, and vertex the point D; for they are contained by the same planes: therefore the pyramid of which the base is the triangle ABD, and vertex the point C, is equal to the pyramid, the base of which is the triangle EBC, and vertex the point D. Again, because, FCBE is a parallelogram of which the diagonal is CE, the triangle ECF is equal to the triangle ECB (*a*); therefore the pyramid of which the base is the triangle ECB, and vertex the point D, is equal to the pyramid the base of which is the triangle ECF, and vertex the point D: but the pyramid of which the base is the triangle ECB, and vertex the point D, has been proved equal to the pyramid of which the base is the triangle ABD, and vertex the point C: therefore *the prism* ABCDEF *is divided into three equal pyramids having triangular bases,* viz. *into the pyramids* ABDC, EBDC, ECFD. And because the pyramid of which the base is the triangle ABD, and vertex the point C, is the same with the pyramid of which the base is the triangle ABC, and vertex the point D, for they are contained by the same planes; and that the pyramid of which the base is the triangle ABD, and vertex the point C, has been demonstrated to be a third part of the prism, the base of which is the triangle ABC, and DEF the opposite triangle; therefore, *the pyramid of which the base is the triangle* ABC, *and vertex the point* D, *is the third part of the prism which has the same base,* viz. *the triangle* ABC, *and* DEF *its opposite triangle.*

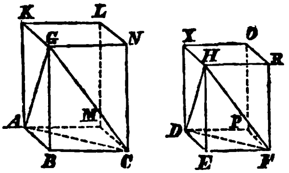
Corollary 1. From this it is manifest that every pyramid is the third part of a prism which has the same base, and is of an equal altitude with it: for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

Corollary 2. Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are to one another as their bases (*c*).

PROPOSITION VIII.

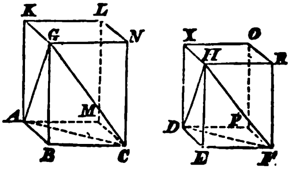
THEOREM.—*Similar pyramids having triangular bases* are one to another in the triplicate ratio of that of their homologous sides.

DEMONSTRATION. Let the pyramids having the triangles ABC, DEF for their bases, and the points G, H for their vertices, be similar and similarly situated: the pyramid ABCG shall have to the pyramid DEFH, the triplicate ratio of that which the side BC has to the homologous side EF.



(a) XI. Def. 11.
(b) VI. Def. 1.
(c) XI. 24.
(d) XI. B.
(e) XI. 33.
(f) V. 15.

Complete the parallelograms ABCM, GBCN, ABGK, *and the solid parallelopiped* BGML *contained by these planes and those opposite to them; and, in like manner, complete the solid parallelopiped* EHPO *contained by the three parallelograms* DEFP, HEFR, DEHX, *and those opposite to them.* And because the pyramid ABCG is similar to the pyramid DEFH, the angle ABC is equal to the angle DEF (*a*), and the angle GBC to the angle HEF, and ABG to DEH: and AB is to BC as DE is to EF (*b*); that is, the sides about the equal angles are proportionals: wherefore the parallelogram BM is similar to EP: for the same reason, the parallelogram BN is similar to ER, and BK to EX: therefore the three parallelograms BM, BN, BK are similar to the three EP, ER, EX: but the three BM, BN, BK are equal and similar to the three which are opposite to them (*c*), and the three EP, ER, EX equal and similar to the three opposite to them: wherefore the solids BGML, EHPO are contained by the same number of similar planes: and their solid angles are equal (*d*); and therefore the solid BGML is similar to the solid EHPO (*a*): but similar solid parallelopipeds have the triplicate ratio of that which their homologous sides have (*e*): therefore the solid BGML has to the solid EHPO, the triplicate ratio of that which the side BC has to the homologous side EF: but as the solid BGML is to the solid EHPO, so is the pyramid ABCG to the pyramid DEFH (*f*); because the pyramids are the sixth part of the solids, since the prism, which is the half (g) of the solid parallelopiped, is triple of the pyramid (h): wherefore, likewise, *the pyramid* ABCG *has to the pyramid* DEFH, *the triplicate ratio of that which* BC *has to the homologous side* EF.



(g) XI. 28.
(h) XII. 7.
(i) V. 12.

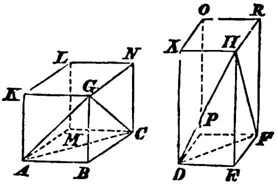
COROLLARY. From this it is evident, that similar pyramids, which have multangular bases, are likewise to one another in the triplicate ratio of their homologous sides: for they may be divided into similar pyramids having triangular bases, because the similar polygons which are their bases, may be divided into the same number of similar triangles homologous to the whole polygons: therefore, as one of the triangular pyramids in the first multangular pyramid is to one of the triangular pyramids in the other (i), so are all the triangular pyramids in the first to all the triangular pyramids in the other; that is, so is the first multangular pyramid to the other: but one triangular pyramid is to its similar triangular pyramid in the triplicate ratio of their homologous sides; and therefore the first multangular pyramid has to the other the triplicate ratio of that which one of the sides of the first has to the homologous side of the other.

PROPOSITION IX.

THEOREM [1.]—*The bases and altitudes of equal pyramids having triangular bases* are reciprocally proportional; [2.] *and triangular pyramids, of which the bases and altitudes are reciprocally proportional,* are equal to one another.

DEMONSTRATION [1.] Let the pyramids of which the triangles ABC, DEF are the bases, and which have their vertices in the points G, H, be equal to one another: the bases and altitudes of the pyramids ABCG, DEFH shall be reciprocally proportional, viz. the base ABC shall be to the base DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelogram AC, AG, GC, DF, DH, HF; *and the solid parallelopipeds* BGML, EHPO, *contained by these planes, and those which are opposite to them.* And because the pyramid ABCG is equal to the pyramid DEFH, and that the solid BGML is sextuple of the pyramid ABCG (a), and the solid EHPO sextuple of



the pyramid DEFH; therefore the solid BGML is equal to the solid EHPO (*b*): but the bases and altitudes of equal solid parallelopipeds are reciprocally proportional (*c*); therefore, as the base BM is to the base EP, so is the altitude of the solid EHPO to the altitude of the solid BGML: but as the base BM is to the base EP, so is the triangle ABC to the triangle DEF (*d*); therefore as the triangle ABC to the triangle DEF, so is the altitude of the solid EHPO to the altitude of the solid BGML: but the altitude of the solid EHPO is the same with the altitude of the pyramid DEFH; and the altitude of the solid BGML is the same with the altitude of the pyramid ABCG; therefore, as the base ABC to the base DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: wherefore, *the bases and altitudes of the pyramids* ABCG, DEFH, *are reciprocally proportional.*

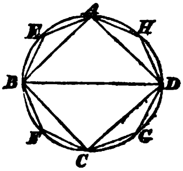
(*a*) XI. 28, and XII. 7.
(*b*) V. Ax. 1.
(*c*) XI. 34.
(*d*) V. 15.
(*e*) V. Ax. 2.

[2.] Again, let the bases and altitudes of the pyramids ABCG, DEFH, be reciprocally proportional, viz. the base ABC be to the base DEF, as the altitude of the pyramid DEFH is to the altitude of the pyramid ABCG: the pyramid ABCG shall be equal to the pyramid DEFH.

The same construction being made; because as the base ABC is to the base DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG; and as the base ABC is to the base DEF, so is the parallelogram BM to the parallelogram EP: therefore the parallelogram BM is to EP, as the altitude of the pyramid DEFH is to the altitude of the pyramid ABCG: but the altitude of the pyramid DEFH is the same with the altitude of the solid parallelopiped EHPO; and the altitude of the pyramid ABCG is the same with the altitude of the solid parallelopiped BGML: therefore as the base BM is to the base EP, so is the altitude of the solid parallelopiped EHPO to the altitude of the solid parallelopiped BGML: but solid parallelopipeds having their bases and altitudes reciprocally proportional, are equal to one another (*c*); therefore the solid parallelopiped BGML is equal to the solid parallelopiped EHPO: and the pyramid ABCG is the sixth part of the solid BGML, and the pyramid DEFH is the sixth part of the solid EHPO; therefore *the pyramid* ABCG *is equal to the pyramid* DEFH (*e*).

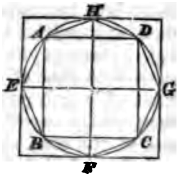
PROPOSITION X.

THEOREM.—*Every cone* is the third part of a cylinder which has the same base (ABCD), and is of an equal altitude with it.



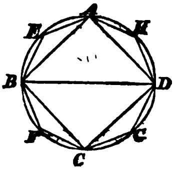
(a) XII. 2.
(b) XI. 32.

DEMONSTRATION. If the cylinder be not triple of the cone, it must either be greater than the triple, or less than it. First, let it be greater than the triple; and inscribe the square ABCD in the circle: this square is greater than the half of the circle ABCD (*a*). Upon the square ABCD, erect a prism of the same altitude with the cylinder; this prism shall be greater than half of the cylinder: for let a square be described about the circle, and let a prism be erected upon the square, of the same altitude with the cylinder; then the inscribed square is half of that circumscribed; and upon these square bases are erected solid parallelopipeds, viz. the prisms of the same altitude; therefore the prism upon the square ABCD is the half of the prism upon the square described about the circle; because they are to one another as their bases (*b*): and the cylinder is less than the prism upon the square described about the circle ABCD; therefore the prism upon the square ABCD of the same altitude with the cylinder, is greater than half of the cylinder. *Bisect the circumferences* AB, BC, CD, DA, *in the points* E, F, G, H; *and join* AE, EB, BF, FC, CG, GD, DH, HA: then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the segment of the circle in which it stands, as was shown in Prop. II. of this book. *Erect prisms upon each of these triangles, of the same altitude with the cylinder;* each of these prisms shall be greater than half of the segment of the cylinder in which it is; *because, if through the points* E, F, G, H *parallels be drawn to* AB, BC, CD, DA, *and parallelograms be completed upon the same* AB, BC, CD, DA, *and solid parallelopipeds be erected upon the parallelograms;* the prisms upon the triangles AEB, BFC, CGD, DHA, are the halves of the solid parallelopipeds (*c*); and the segments of the cylinder which are upon the segments of the circle cut off by AB, BC, CD, DA are less than the solid parallelopipeds which contain them; therefore the prisms upon the triangles AEB, BFC, CGD, DHA are greater than half of the segments of the cylinder in which they are: therefore, if each of the circumferences be divided into two equal parts, and straight lines be drawn from the points of division to the extremities of the circumferences, and

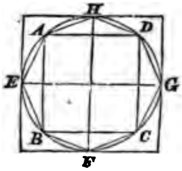


upon the triangles thus made, prisms be erected of the same altitude with the cylinder, and so on, there must at length remain some segments of the cylinder which together are less than the excess of the cylinder above the triple of the cone (*d*): let them be those upon the segments of the circle, AE, EB, BF, FC, CG, GD, DH, HA; therefore the rest of the cylinder, that is, the prism of which the base is the polygon AEBFCGDH, and of which the altitude is the same with that of the cylinder, is greater than the triple of the cone: but this prism is triple of the pyramid upon the same base (*e*), of which the vertex is the same with the vertex of the cone; therefore the pyramid upon the base AEBFCGDH, having the same vertex with the cone, is greater than the cone of which the base is the circle ABCD: but it is also less, for the pyramid is contained within the cone; which is impossible; therefore the cylinder is not greater than the triple of the cone.

(*c*) XII. 7, cor. 2.
(*d*) XII. Lemma 1.
(*e*) XII. 7, cor. 1.



Nor can the cylinder be less than the triple of the cone. Let it be less, if possible; therefore, inversely, the cone is greater than the third part of the cylinder. *In the circle* ABCD, *inscribe a square:* this square is greater than the half of the circle: *and upon the square* ABCD *erect a pyramid* having the same vertex with the cone; this pyramid is greater than the half of the cone; because, as was before demonstrated, if a square be described about the circle, the square ABCD is the half of it: and if upon these squares there be erected solid parallelopipeds of the same altitude with the cone, which are also prisms, the prism upon the square ABCD is the half of that which is upon the square described about the circle; for they are to one another as their bases (*b*); as are also the third parts of them: therefore the pyramid, the base of which is the square ABCD, is half of the pyramid upon the square described about the circle: but this last pyramid is greater than the cone which it contains; therefore the pyramid upon the square ABCD, having the same vertex with the cone, is greater than the half of the cone. *Bisect the circumferences* AB, BC, CD, DA, *in the points* E, F, G, H, *and join* AE, EB, BF, FC, CG, GD, DH, HA: therefore each of the triangles AEB, BFC, CGD, DHA is greater than half of the segment of the circle in which it is: upon each of these triangles erect pyramids having the same vertex with the cone: therefore each of those pyramids is greater than the half of the segment of the cone in which it is, as was before demonstrated of the prisms and segments of the cylinder: and thus dividing each of the circumferences into two equal parts, and joining the points of division and their extremities by straight lines, and upon the triangles erecting pyramids having their vertices the same with that of the cone, and so on, there must at length remain some segments of the cone, which together are less than the excess of the cone above the third part of the cylinder (*d*): let these be the segments upon AE, EB, BF, FC, CG, GD, DH, HA: therefore the rest of the cone, that is, the pyramid of which the base is the polygon AEBFCGDH, and of which the vertex is the same with that of the cone, is greater than the third part of the cylinder: but this pyramid is the third part of the prism upon the same base AEBFCGDH, and of the same altitude with the cylinder; therefore this prism is greater than the cylinder of which the base is the circle ABCD: but it is also less, for it is contained within the cylinder; which is impossible: therefore the cylinder is not less than the triple of the cone. And it has been demonstrated, that neither is it greater than the triple; therefore the cylinder is triple of the cone, or, *the cone is the third part of the cylinder.*



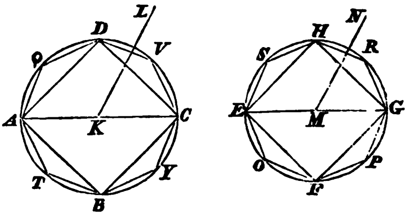
(*d*) XII. Lemma 1.

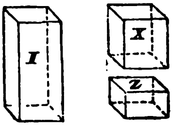
PROPOSITION XI.

Theorem.—*If cones and cylinders are of the same altitude,* they are to one another as their bases.

Demonstration. Let the cones and cylinders, of which the bases are the circles ABCD, EFGH, and the axes KL, MN, and AC, EG, the diameters of their bases, be of the same altitude: as the circle ABCD is to the circle EFGH, so shall the cone AL be to the cone EN.

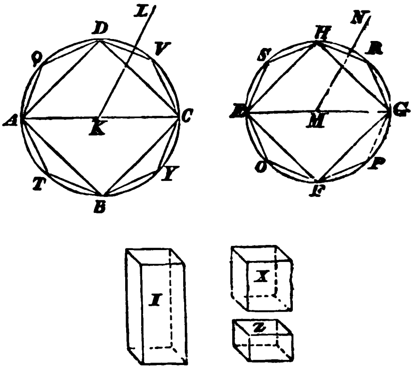
If it be not so, the circle ABCD must be to the circle EFGH, as the cone AL to some solid either less than the cone EN, or greater than it. First, let it be to a solid less than EN, viz. to the solid X; and let Z be the solid which is equal to the excess of the cone EN above the solid X; therefore the cone EN is equal to the solids X, Z together. *In the circle* EFGH, *inscribe the square* EFGH; therefore this square is greater than the half of the circle: *upon the square* EFGH, *erect a pyramid of the same altitude with the cone;* this pyramid shall be greater than half of the cone: for, if a square be described about the circle, and a pyramid be erected upon it, having the same vertex with the cone, the pyramid inscribed in the cone is half of the pyramid circumscribed about it, because they are to one another as their bases (*a*): but the cone is less than the circumscribed pyramid; therefore the pyramid of which the base is the square EFGH, and its vertex





(a) XII. 6.
(b) XII. Lemma 1.
(c) XII. 1.
(d) XII. 2.
(e) V. 11.

the same with that of the cone, is greater than half of the cone. *Divide the circumferences* EF, FG, GH, HE, *each into two equal parts in the points* O, P, R, S, *and join* EO, OF, FP, PG, GR, RH, HS, SE: therefore each of the triangles EOF, FPG, GRH, HSE, is greater than half of the segment of the circle in which it is: *upon each of these triangles, erect a pyramid having the same vertex with the cone;* each of these pyramids is greater than the half of the segment of the cone in which it is: and thus dividing each of these circumferences into two equal parts, and, from the points of division drawing straight lines to the extremities of the circumferences, and upon each of the triangles thus made, erecting pyramids having the same vertex with the cone, and so on, there must at length remain some segments of the cone which are together less than the solid Z (*b*); let these be the segments upon EO, OF, FP, PG, GR, RH, HS, SE: therefore the remainder of the cone, viz. the pyramid of which the base is the polygon EOFPGRHS, and its vertex the same with that of the cone, is greater than the solid X. *In the circle* ABCD, *inscribe the polygon* ATBYCVDQ *similar to the polygon* EOFPGRHS, *and upon it erect a pyramid having the same vertex with the cone* AL: and because as the square on AC is to the square on EG, so is the polygon ATBYCVDQ to the polygon EOFPGRHS (*c*); and as the square on AC is to the square on EG, so is the circle ABCD to the circle EFGH (*d*); therefore the circle ABCD is to the circle EFGH, as the polygon ATBYCVDQ to the polygon EOFPGRHS (*e*): but as the circle ABCD is to the circle EFGH, so is the cone AL to the solid X; and as the polygon ATBYCVDQ is to the polygon EOFPGRHS, so is (*a*) the pyramid of which the base is the first of these polygons, and vertex L, to the pyramid of which the base is the other polygon, and its vertex N: therefore, as the cone AL is to the solid X, so is the pyramid of which the base is the polygon ATBYCVDQ, and vertex L, to the pyramid the base of which is the polygyon EOFPGRHS, and vertex N: but the cone AL is greater than the pyramid contained in it; therefore the solid X is greater than the pyramid in the cone EN (*f*): but it is less, as was shown; which is absurd: therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any solid which is less than the cone EN. In the same manner it may be demonstrated, that the circle EFGH is not to the circle ABCD, as the cone EN to any solid less than the cone AL. Nor can the circle ABCD be to the circle EFGH, as the cone AL, to any solid greater than the cone EN. For if it be possible, let it be so to the solid I, which is greater than the cone EN: therefore, by inversion, as the circle EFGH is to the circle ABCD, so is the solid I to the cone AL: but as the solid I is to the cone AL, so is the cone EN to some solid, which must be less than the cone AL (*f*), because the solid I is greater than the cone EN; therefore, as the circle EFGH is to the circle ABCD, so is the cone EN to a solid less than the cone AL, which was shown to be impossible; therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any solid greater than the cone EN. And it has been demonstrated, that neither is the circle ABCD to the circle EFGH, as the cone AL to any solid less than the cone EN; therefore the circle ABCD is to the circle EFGH, as the cone AL is to the cone EN: but as the cone is to the cone, so is the cylinder to the cylinder (*g*), because the cylinders are triple of the cones, each of each (*h*): therefore, *as the circle* ABCD *is to the circle* EFGH, *so are the cylinders upon them of the same altitude.*

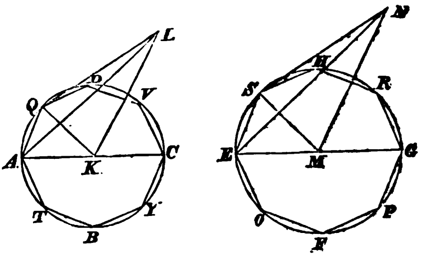


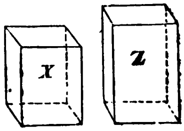
(*f*) V. 14.
(*g*) V. 15.
(*h*) XII. 10.

PROPOSITION XII.

THEOREM.—*If cones and cylinders are similar*, they have to one another, the triplicate ratio of that which the diameters of their bases have.

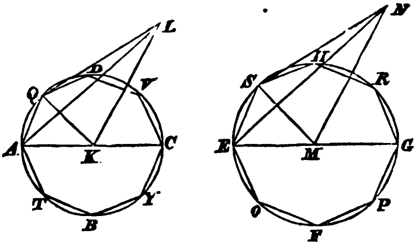
DEMONSTRATION. Let the cones and cylinders of which the bases are the circles ABCD, EFGH, and the diameters of the bases AC, EG, and KL, MN the axes of the cones or cylinders, be similar: the cone of which the base is the circle ABCD and vertex the point L, shall have to the cone of which the base is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG.

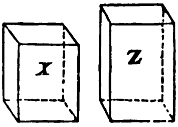




For if the cone ABCDL has not to the cone EFGHN, the triplicate ratio of that which AC has to EG, the cone ABCDL must have the triplicate of that ratio to some solid which is less or greater than the cone EFGHN. First, let it have it to a less, viz. to the solid X. *Make the same construction as in the preceding proposition*, and it may be demonstrated in the very same way as in that proposition, that the pyramid of which the base is the polygon EOFPGRHS, and vertex N, is greater than the solid X. *Inscribe also in the circle* ABCD, *the polygon* ATBYCVDQ *similar to the polygon* EOFPGRHS, *upon which erect a pyramid having the same vertex with the cone:* and let LAQ be one of the triangles containing the pyramid upon the polygon ATBYCVDQ, the vertex of which is L; and let NES be one of the triangles containing the pyramid upon the polygon EOFPGRHS, of which the vertex is N; and join KQ, MS. Then, because the cone ABCDL is similar to the cone EFGHN, AC is to EG as the axis KL is to the axis MN (*a*); and as AC is to EG, so is AK to EM (*b*); therefore as AK is to EM, so is KL to MN; and alternately, AK is to KL, as EM is to MN: and the right angles AKL, EMN are equal: therefore the sides about these equal angles being proportionals, the triangle AKL is similar to the triangle EMN (*c*). Again, because AK is to KQ, as EM is to MS, and that these sides are about equal angles AKQ, EMS, because these angles are, each of them, the same part of four right angles at the centers K, M, therefore the triangle AKQ is similar to the triangle EMS (*c*). And because it has been shown, that as AK is to KL, so is EM to MN, and that AK is equal to KQ, and EM to MS, therefore as QK is to KL, so is SM to MN; and therefore the

(*a*) XI. 24.
(*b*) V. 15.
(*c*) VI. 6.



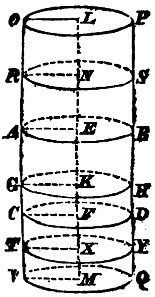


(*d*) V. 22.
(*e*) VI. 5.
(*f*) XI. B.
(*g*) XII. 8.
(*h*) V. 12.
(*i*) V. 14.
(*k*) XII. 10.

sides about the right angles QKL, SMN, being proportionals, the triangle LKQ is similar to the triangle NMS. And because of the similarity of the triangles AKL, EMN, as LA is to AK, so is NE to EM; and by the similarity of the triangles AKQ, EMS, as KA is to AQ, so is ME to ES: therefore, *ex æquali,* LA is to AQ, as NE to ES (*d*). Again, because of the similarity of the triangles LQK, NSM, as LQ to QK, so is NS to SM; and from the similarity of the triangles, KAQ, MES, as KQ is to QA, so is MS to SE: therefore, *ex æquali,* LQ is to QA, as NS is to SE (*d*): and it was proved, that QA is to AL, as SE is to EN: therefore again, *ex æquali,* as QL is to LA, so is SN to NE: wherefore the triangles LQA, NSE, having the sides about all their angles proportionals, are equiangular and similar to one another (*e*): and therefore the pyramid of which the base is the triangle AKQ, and vertex L, is similar to the pyramid the base of which is the triangle EMS, and vertex N, because their solid angles are equal to one another (*f*); and they are contained by the same number of similar planes: but similar pyramids which have triangular bases, have to one another the triplicate ratio of that which their homologous sides have (*g*); therefore the pyramid AKQL has to the pyramid EMSN, the triplicate ratio of that which AK has to EM. In the same manner, *if straight lines be drawn from the points* D, V, C, Y, B, T, *to* K, *and from the points* H, R, G, P, F, O, *to* M, *and pyramids be erected upon the triangles, having the same vertices with the cones,* it may be demonstrated, that each pyramid in the first cone has to each in the other, taking them in the same order, the triplicate ratio of that which the side AK has to the side EM: that is, which AC has to EG: but as one antecedent is to its consequent, so are all the antecedents to all the consequents (*h*); therefore as the pyramid AKQL is to the pyramid EMSN, so is the whole pyramid the base of which is the polygon DQATBYCV, and vertex L, to the whole pyramid of which the base is the polygon HSEOFPGR, and vertex N: wherefore also the first of these two last-named pyramids has to the other the triplicate ratio of that which AC has to EG; but, by the hypothesis, the cone of which the base is the circle ABCD, and vertex L, has to the solid X, the triplicate ratio of that which AC has to EG; therefore as the cone of which the base is the circle ABCD, and vertex L, is to the solid X, so is the pyramid the base of which is the polygon DQATBYCV, and vertex L, to the pyramid the base of which is the polygon HSEOFPGR, and vertex N: but the said cone is greater than the pyramid contained in it: therefore the solid X is greater than the pyramid (*i*), the base of which is the polygon HSEOFPGR, and vertex N: but it is also less; which is impossible: therefore the cone, of which the base is the circle ABCD, and vertex L, has not to any solid which is less than the cone of which the base is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG. In the same manner it may be demonstrated, that neither has the cone EFGHN to any solid which is less than the cone ABCDL, the triplicate ratio of that which EG has to AC. Nor can the cone ABCDL have to any solid which is greater than the cone EFGHN, the triplicate ratio of that which AC has to EG. For if it be possible, let it have it to a greater, viz. to the solid Z: therefore, inversely, the solid Z has to the cone ABCDL, the triplicate ratio of that which EG has to AC: but as the solid Z is to the cone ABCDL, so is the cone EFGHN to some solid, which must be less than the cone ABCDL (*i*), because the solid Z is greater than the cone EFGHN; therefore the cone EFGHN has to a solid which is less than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonstrated to be impossible: therefore the cone ABCDL has not to any solid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG: and it was demonstrated that it could not have that ratio to any solid less than the cone EFGHN: therefore the cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG; but as the cone is to the cone, so is the cylinder to the cylinder (*b*); for every cone is the third part of the cylinder upon the same base, and of the same altitude (*k*): therefore *also the cylinder has to the cylinder, the triplicate ratio of that which* AC *has to* EG.

PROPOSITION XIII.

THEOREM.—*If a cylinder be cut by a plane parallel to its opposite planes or bases*, it divides the cylinder into two cylinders, one of which is to the other, as the axis of the first is to the axis of the other.



(a) XI. 16.
(b) I. Def. 15.
(c) XII. 11.
(d) V. Def. 5.

Demonstration. Let the cylinder AD be cut by the plane GH parallel to the opposite planes AB, CD, meeting the axis EF in the point K: and let the line GH be the common section of the plane GH, and the surface of the cylinder AD. Let AEFC be the parallelogram in any position of it, by the revolution of which about the straight line EF, the cylinder AD is described; and let GK be the common section of the plane GH, and the plane AEFC. And because the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common sections with it, are parallel (a); wherefore AK is a parallelogram, and GK equal to EA, the straight line from the center of the circle AB: for the same reason, each of the straight lines drawn from the point K to the line GH, may be proved to be equal to those which are drawn from the center of the circle AB to its circumference, and are therefore all equal to one another; therefore the line GH is the circumference of a circle of which the center is the point K (b): therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the same which would be described by the revolution of the parallelograms AK, GF, about the straight lines EK, KF: and it is to be shown, that the cylinder AH is to the cylinder HC, as the axis EK is to the axis KF.

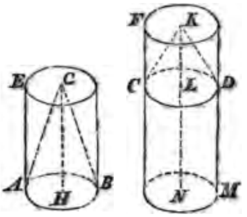
Produce the axis EF *both ways: and take any number of straight lines* EN, NL, *each equal to* EK; *and any number* FX, XM, *each equal to* FK; *and let planes parallel to* AB, CD, *pass through the points* L, N, X, M: therefore the common sections of these planes with the cylinder produced, are circles, the centers of which are the points L, N, X, M, as was proved of the plane GH; and these planes cut off the cylinders PR, RB, DT, TQ. And because the axes LN, NE, EK are all equal, therefore the cylinders PR, RB, BG are to one another as their bases (c): but their bases are equal, and therefore the cylinders PR, RB, BG are equal: and because the axes LN, NE, EK are equal to one another, as also the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore whatever multiple the axis KL is of the axis KE, the same multiple is the cylinder PG of the cylinder GB: for the same reason, whatever multiple the axis MK is of the axis KF, the same multiple is the cylinder QG of the cylinder GD: and if the axis KL be equal to the axis KM, the cylinder PG is equal to the cylinder GQ; and if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if less, less: therefore, since there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD; and that of the axis EK and cylinder

BG there have been taken any equimultiples whatever, viz. the axis KL and cylinder PG, and of the axis KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ; and since it has been demonstrated, that if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if equal, equal; and if less, less; therefore (*d*), *as the axis* EK *is to the axis* KF, *so is the cylinder* BG *to the cylinder* GD.

PROPOSITION XIV.

THEOREM.—*If cones and cylinders are upon equal bases,* they are to one another as their altitudes.

DEMONSTRATION. Let the cylinders EB, FD be upon the equal bases AB, CD; as the cylinder EB is to the cylinder FD, so shall the axis GH be to the axis KL.



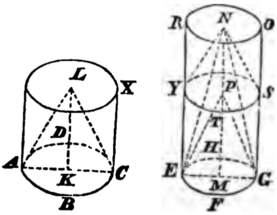
Produce the axis KL *to the point* N, *and make* LN *equal to the axis* GH; *and let* CM *be a cylinder of which the base is* CD, *and axis* LN. Then, because the cylinders EB, CM have the same altitude, they are to one another as their bases (*a*): but their bases are equal, therefore also the cylinders EB, CM are equal: and because the cylinder FM is cut by the plane CD parallel to its opposite planes, as the cylinder CM is to the cylinder FD, so is the axis LN to the axis KL (*b*): but the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH; therefore as the cylinder EB is to the cylinder FD, so is the axis GH to the axis KL: and as the cylinder EB is to the cylinder FD, so is the cone ABG to the cone CDK (*c*), because the cylinders are triple of the cones (*d*): therefore *also the axis* GH *is to the axis* KL, *as the cone* ABG *is to the cone* CDK, *and as the cylinder* EB *is to the cylinder* FD.

(*a*) XII. 11.
(*b*) XII. 13.
(*c*) V. 15.
(*d*) XII. 10.

PROPOSITION XV.

THEOREM [1.]—*If cones and cylinders are equal,* their bases and altitudes are reciprocally proportional; [2.] *and if the bases and altitudes be reciprocally proportional,* the cones and cylinders are equal to one another.

Demonstration [1]. Let the circles ABCD, EFGH, the diameters of which are AC, EG, be the bases, and KL, MN the axes, as also the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders: the bases and altitudes of the cylinders AX, EO shall be reciprocally proportional; that is, as the base ABCD is to the base EFGH, so shall the altitude MN be to the altitude KL.



(a) XII. 11.
(b) V. A.
(c) V. 7.
(d) XII. 13.

Either the altitude MN is equal to the altitude KL, or these altitudes are not equal. First let them be equal; and the cylinders AX, EO being also equal, and cones and cylinders of the same altitude being to one another as their bases (a), therefore the base ABCD is equal to the base EFGH (b); and as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL. But let the altitudes KL, MN be unequal, and MN the greater of the two, *and from* MN *take* MP *equal to* KL, *and through the point* P *cut the cylinder* EO *by the plane* TYS, *parallel to the opposite planes of the circles* EFGH, RO: therefore the common section of the plane TYS and the cylinder EO is a circle, and consequently ES is a cylinder, the base of which is the circle EFGH, and altitude MP: and because the cylinder AX is equal to the cylinder EO, as AX is to the cylinder ES, so is the cylinder EO to the same ES (c): but as the cylinder AX is to the cylinder ES, so is the base ABCD to the base EFGH (a); for the cylinders AX, ES are of the same altitude; and as the cylinder EO is to the cylinder ES, so is the altitude MN to the altitude MP (d), because the cylinder EO is cut by the plane TYS parallel to its opposite planes; therefore as the base ABCD is to the base EFGH, so is the altitude MN to the altitude MP: but MP is equal to the altitude KL: wherefore as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL; that is, *the bases and altitudes of the equal cylinders* AX, EO *are reciprocally proportional.*

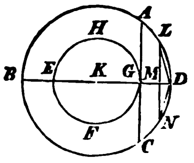
[2.] But let the bases and altitudes of the cylinders AX, EO be reciprocally proportional, viz. the base ABCD to the base EFGH, as the altitude MN is to the altitude KL: the cylinder AX shall be equal to the cylinder EO.

First, let the base ABCD be equal to the base EFGH: then, because as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL; MN is equal to KL (b); and *therefore the cylinder* AX *is equal to the cylinder* EO (a).

But let the bases ABCD, EFGH be unequal, and let ABCD be the greater; and because as ABCD is to the base EFGH, so is the altitude MN to the altitude KL; therefore MN is greater than KL (*b*). *Then, the same construction being made as before,* because as the base ABCD is to the base EFGH, so is the altitude MN to the altitude KL; and because the altitude KL is equal to the altitude MP; therefore the base ABCD is to the base EFGH, as the cylinder AX is to the cylinder ES (*a*); and as the altitude MN is to the altitude MP or KL, so is the cylinder EO to the cylinder ES: therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the same ES: whence, the cylinder AX *is equal to the cylinder* EO: *and the same reasoning holds in cones.*

PROPOSITION XVI.

Problem.—In the greater of two given *circles* (ABCD, EFGH) *that have the same center* (K), to inscribe a polygon of an even number of equal sides, that shall not meet the lesser circle (EFGH).



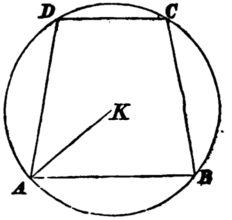
(*a*) III. 16, Cor.
(*b*) XII. Lemma 1.

Solution. *Through the center* K *draw the straight line* BD, *and from the point* G, *where it meets the circumference of the lesser circle, draw* GA *at right angles to* BD, *and produce it to* C; *therefore* AC *touches the circle* EFGH (*a*): *then, if the circumference* BAD *be bisected, and the half of it be again bisected, and so on, there must at length remain a circumference less than* AD (*b*): *let this be* LD; *and from the point* L *draw* LM *perpendicular to* BD, *and produce it to* N; *and join* LD, DN; therefore LD is equal to DN: and because LN is parallel to AC, and that AC touches the circle EFGH; therefore LN does not meet the circle EFGH; and much less shall the straight lines LD, DN meet the circle EFGH; *so that if straight lines equal to* LD *be applied in the circle* ABCD *from the point* L *around to* N, *there shall be inscribed in the circle a polygon of an even number of equal sides not meeting the lesser circle.*

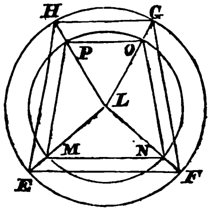
LEMMA II.

THEOREM.—*If two trapeziums* ABCD, EFGH *be inscribed in the circles, the centers of which are the points* K, L; *and if the sides* AB, DC *be parallel, as also* EF, HG; *and the other four sides* AD, BC, EH, FG *be all equal to one another, but the side* AB *greater than* EF, *and* DC *greater than* HG; the straight line KA, from the center of the circle in which the greater sides are, is greater than the straight line LE, drawn from the center to the circumference of the other circle.

DEMONSTRATION. If it be possible, let KA be not greater than LE; then KA must be either equal to it, or less than it. First, let KA be equal to LE: therefore, because in two equal circles, AD, BC in the one, are equal to EH, FG in the other, the circumferences AD, BC are equal to the circumferences EH, FG (*a*); but because the straight lines AB, DC are respectively greater than EF, GH, the circumferences AB, DC are greater than EF, HG; therefore the whole circumference ABCD is greater than the whole EFGH; but it is also equal to it, which is impossible; therefore the straight line KA is not equal to LE.



(*a*) III. 28.



(*b*) VI. 2.

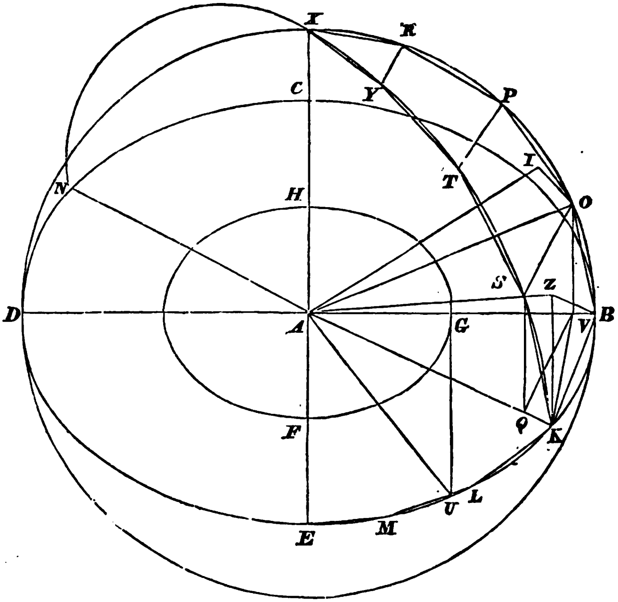
But let KA be less than LE, *and make* LM *equal to* KA; *and from the center* L *and distance* LM *describe the circle* MNOP, *meeting the straight lines* LE, LF, LG, LH, *in* M, N, O, P; *and join* MN, NO, OP, PM which are respectively parallel to and less than EF, FG, GH, HE (*b*): then because EH is greater than MP, AD is greater than MP; and the circles ABCD, MNOP are equal; therefore the circumference AD is greater than MP: for the same reason, the circumference BC is greater than NO: and because the straight line AB is greater than EF, which is greater than MN, much more is AB greater than MN; therefore the circumference AB is greater than MN: and for the same reason, the circumference DC is greater than PO; therefore the whole circumference ABCD is greater than the whole MNOP: but it is likewise equal to it, which is impossible; therefore KA is not less than LE: nor is it equal to it; therefore, *the straight line* KA *must be greater than* LE.

Corollary. And if there be an isosceles triangle, the sides of which are equal to AD, BC, but its base less than AB the greater of the two sides AB, DC; the straight line KA may, in the same manner, be demonstrated to be greater than the straight line drawn from the center to the circumference of the circle described about the triangle.

PROPOSITION XVII.

Problem.—In the greater of two given *spheres which have the same center* (A), to inscribe a solid polyhedron, the superficies of which shall not meet the lesser sphere.

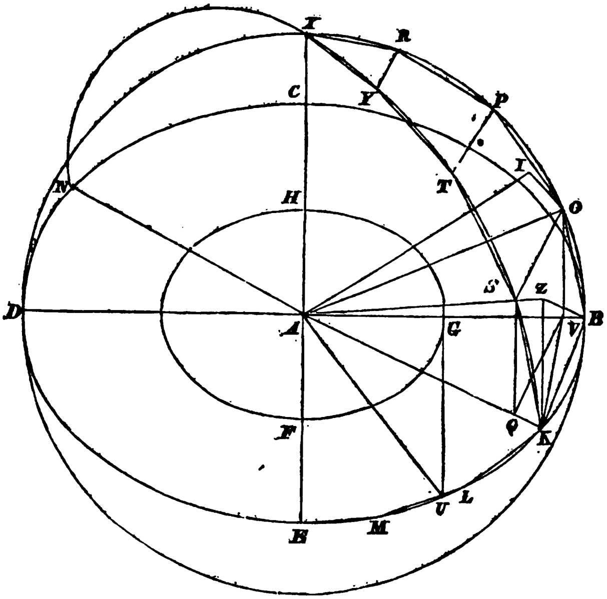
Solution. *Let the spheres be cut by a plane passing through the center;* the common sections of it with the spheres shall be circles, because the sphere is described by the revolution of a semicircle about the diameter remaining immovable; so that in whatever position the semicircle be conceived, the common section of the plane in which it is with the superficies of the sphere is the circumference of a circle; and this is a great circle of the sphere, because the diameter of the sphere, which is likewise the diameter of the circle, is greater than any straight line in the circle or sphere (*a*). Let then the circle made by the section of the plane with the greater sphere be BCDE, and with the lesser sphere be FGH: *and draw the two diameters* BD, CE *at right angles to one another; and in* BCDE, *the greater of the two circles, inscribe a polygon of an even number of equal sides not meeting the lesser circle* FGH (*b*); *and let its sides in* BE, *the fourth part of the circle, be* BK, KL, LM, ME; *join* KA, *and produce it to* N; *and from* A *draw* AX *at right angles to the plane of the circle* BCDE (*c*), *meeting the superficies of the sphere in the point* X: *and let planes pass through* AX, *and each of the straight lines* BD, KN, *which, from* *what has been said, shall produce great circles on the superficies of the sphere, aad let* BXD, KXN *be the semicircles thus made upon the*



(*a*) III. 15.
(*b*) XII. 16.
(*c*) XI. 12.
(*d*) XI. 18.
(*e*) XI. Def. 4.
(*f*) I. 26.
(*g*) VI. 2.
(*h*) XI. 6.
(*i*) I. 33.
(*k*) XI. 9.
(*l*) XI. 2.
(*m*) XI. 11.
(*n*) I. 47.

diameters BD, KN: therefore, because XA is at right angles to the plane of the circle BCDE, every plane which passes through XA is at right angles to the plane of the circle BCDE (*d*); wherefore the semicircles BXD, KXN are at right angles to that plane: and because the semicircles BED, BXD, KXN upon the equal diameters BD, KN are equal to one another, their halves BE, BX, KX are equal to one another; therefore as many sides of the polygon as are in BE, so many are there in BX, KX, equal to the sides BK, KL, LM, ME: *let these polygons be described, and their sides be* BO, OP, PR, RX; KS, ST, TY, YX; *and join* OS, PT, RY; *and from the points* O, S *draw* OV, SQ *perpendiculars to* AB, AK: and because the plane BOXD is at right angles to the plane BCDE, and in one of them BOXD, OV is drawn perpendicular to AB the common section of the planes, therefore OV is perpendicular to the plane BCDE (*e*): for the same reason, SQ is perpendicular to the same plane, because the plane KSXN is at right angles to the plane BCDE. *Join* VQ, and because in the equal semicircles BXD, KXN the circumferences BO, KS are equal, and OV, SQ are perpendicular to their diameters, therefore OV is equal to SQ (*f*), and BV equal to KQ: but the whole BA is equal to the whole KA, therefore the remainder VA is equal to the remainder QA: therefore as BV is to VA, so is KQ to QA; wherefore VQ is parallel to BK (*g*): and because OV, SQ are each of them at right angles to the plane of the circle, BCDE, OV is parallel to SQ (*h*); and it has been proved that it is also equal to it; therefore QV, SO are equal and parallel (*i*); and because QV is parallel to SO, and also to KB, OS is parallel to BK (*k*); and therefore BO, KS, which join them, are in the same plane in which these parallels are, and the quadrilateral figure KBOS is in one plane: *and if* PB, TK *be joined, and perpendiculars be drawn from the points* P, T, *to the straight lines* AB, AK, it may be demonstrated, that TP is parallel to KB in the very same way that SO was shown to be parallel to the same KB; wherefore TP is parallel to SO (*k*), and the quadrilateral figure SOPT is in one plane: for the same reason, the quadrilateral TPRY is in one plane: and the figure YRX is also in one plane (*l*): therefore, *if from the points* O, S, P, T, R, Y, *there be drawn straight lines to the point* A, there will be formed a solid polyhedron between the circumferences BX, KX, composed of pyramids, the bases of which are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and of which the common vertex is the point A: *and if the same construction be made upon each of the sides* KL, LM, ME, *as has been done upon* BK, *and the like be done also in the other three quadrants, and in the other hemisphere*, there will be formed a solid polyhedron inscribed in the sphere, composed of pyramids, the bases of which are the aforesaid quadrilateral figures, and the triangle YRX, and those formed in the like manner in the rest of the sphere, the common vertex of them all being the point A.

Also the superficies of this solid polyhedron shall not meet the lesser sphere in which is the circle FGH. For, *from the point* A *draw* AZ *perpendicular to the plane of the quadrilateral* KBOS (*m*), *meeting it in* Z, *and join* BZ, ZK: and because AZ is perpendicular to the plane KBOS, it makes right angles with every straight line meeting it in that plane; therefore AZ is perpendicular to BZ and ZK: and because AB is equal to AK, and that the squares on AZ, ZB are equal to the square on AB, and the squares on AZ, ZK to the square on AK (*n*); therefore the squares on AZ, ZB are equal to the squares on AZ, ZK: take from these equals the square on AZ, and the remaining square on BZ is equal to the remaining square on ZK; and therefore the straight line BZ is equal to ZK: in the like manner it may be demonstrated, that

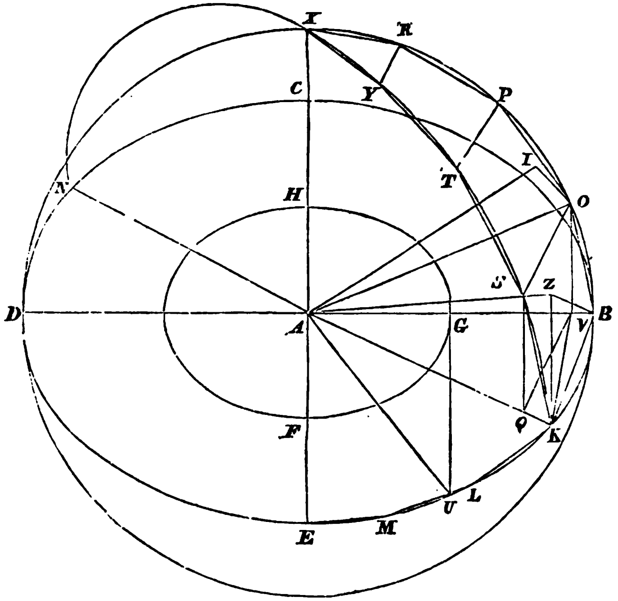


(o) II. 12.
(p) I. 4.
(q) VI. 8.
(r) XII. Lemma 2.
(s) XII. Lemma 2, Cor.

the straight lines drawn from the point Z to the points O, S are equal to BZ or ZK; therefore the circle described from the center Z, and distance ZB, will pass through the points K, O, S, and KBOS will be a quadrilateral figure in the circle: and because KB is greater than QV, and QV equal to SO, therefore KB is greater than SO: but KB is equal to each of the straight lines BO, KS; wherefore each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and these three circumferences, together with a fourth equal to one of them, are greater than the same three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the circumference subtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and consequently the angle BZK at the center is greater than a right angle: and because the angle BZK is obtuse, the square on BK is greater than the squares on BZ, ZK (*o*); that is, greater than twice the square on BZ. *Join* KV: and because (in the triangles KBV, OBV) KB, BV are equal to OB, BV, and that they contain equal angles, the angle KVB is equal to the angle OVB (*p*): and OVB is a right angle; therefore also KBV is a right angle: and because BD is less than twice DV, the rectangle contained by BD, BV is less than twice the rectangle DVB; that is, the square on KB is less than twice the square on KV (*q*): but the square on KB is greater than twice the square on BZ; therefore the square on KV is greater than the square on BZ: and because BA is equal to AK, and that the squares on BZ, ZA are equal together to the square on BA, and the squares on KV, VA to the square on AK; therefore the squares on BZ, ZA are equal to the squares on KV, VA; and of these the square on KV is greater than the square on BZ; therefore the square on VA is less than the square on ZA, and the straight line AZ is greater than VA: much more then is AZ greater than AG; because, in the preceding proposition, it was shown that KV falls without the circle FGH: and AZ is perpendicular to the plane KBOS, and is therefore the shortest of all the straight lines that can be drawn from A, the center of the sphere, to that plane: therefore the plane KBOS does not meet the lesser sphere.

And that the other planes between the quadrants BX, KX fall without the lesser sphere, is thus demonstrated. *From the point* A *draw* AI *perpendicular to the plane of the quadrilateral* SOPT, *and join* IO; and, as was demonstrated of the plane KBOS and the point Z, in the same way it may be shown, that the point I is the center of a circle described about SOPT; and that OS is greater than PT; and PT was shown to be parallel to OS: therefore, because the two trapeziums KBOS, SOPT, inscribed in circles, have their sides BK, OS parallel, as also OS, PT; and their other sides, BO, KS, OP, ST, all equal to one another, and that BK is greater than OS, and OS greater than PT, therefore the straight line ZB is greater than IO (*r*). *Join* AO, which will be equal to AB; and because AIO, AZB are right angles, the squares on AI, IO are equal to the square on AO or of AB; that is, to the squares on AZ, ZB; and the square on ZB is greater than the square on IO, therefore the square on AZ is less than the square on AI; and the straight line AZ less than the straight line AI: and it was proved that AZ is greater than AG; much more then is AI greater than AG: therefore the plane SOPT falls wholly without the lesser sphere. In the same manner it may be demonstrated, that the plane TPRY falls without the same sphere (*s*), as also the triangle YRX. And after the same way it may be demonstrated, that all the planes which contain the solid polyhedron fall without the lesser sphere. Therefore, *in the* *greater of two spheres which have the same center, a solid polyhedron is inscribed, the superficies of which does not meet the lesser sphere.*

SCHOLIUM. The straight line AZ may be demonstrated to be greater than AG otherwise, and in a shorter manner, without the help of Prop. 16, as follows. *From the point* G *draw* GU *at right angles to* AG, *and join* AU. If then the circumference BE be bisected, and its half again bisected, and so on, there will at length be left a circumference less than the circumference which is subtended by a straight line equal to GU, inscribed in the circle BCDE: let this be the circumference KB; therefore the straight line KB is less than GU: and because the angle BZK is obtuse, as was proved in the preceding, therefore BK is greater than BZ: but GU is greater than BK; much more then is GU greater than BZ, and the square on GU than the square on BZ; and AU is equal to AB; therefore the square on AU, that is, the squares on AG, GU are equal to the square on AB, that is, to the squares on AZ, ZB: but the square on BZ is less than the square on GU; therefore the square on AZ is greater than the square on AG, and the straight line AZ consequently greater than the straight line AG.



(*a*) XI. B. (*b*) XI. Def. 11. (*c*) XII. 8, Cor.

COROLLARY. And if in the lesser sphere there be inscribed a solid polyhedron, by drawing straight lines betwixt the points in which the straight lines from the center of the sphere, drawn to all the angles of the solid polyhedron in the greater sphere, meet the superficies of the lesser, in the same order in which are joined the points in which the same lines from the center meet the superficies of the greater sphere, the solid polyhedron in the sphere BCDE shall have to this other solid polyhedron, the triplicate ratio of that which the diameter of the sphere BCDE has to the diameter of the other sphere. For if these two solids be divided into the same number of pyramids, and in the same order, the pyramids shall be similar to one another, each to each: because they have the solid angles at their common vertex, the center of the sphere, the same in each pyramid, and their other solid angles at the bases, equal to one another, each to each (*a*), because they are contained by three plane angles, each equal to each; and the pyramids are contained by the same number of similar planes; and are therefore similar to one another, each to each (*b*): but similar pyramids have to one another, the triplicate ratio of their homologous sides (*c*): therefore the pyramid of which the base is the quadrilateral KBOS, and vertex A, has to the pyramid in the other sphere of the same order, the triplicate ratio of their homologous sides, that is, of that ratio which AB from the center of the greater sphere, has to the straight line from the same center to the superficies of the lesser sphere. And in like manner, each pyramid in the greater sphere has to each of the same order in the lesser, the triplicate ratio of that which AB has to the semi-diameter of the lesser sphere. And as one antecedent is to its consequent, so are all the antecedents to all the consequents. Wherefore, the whole solid polyhedron in the greater sphere has to the whole solid polyhedron in the other, the triplicate ratio of that which AB the semi-diameter of the first has to the semi-diameter of the other; that is, which the diameter BD of the greater has to the diameter of the other sphere.

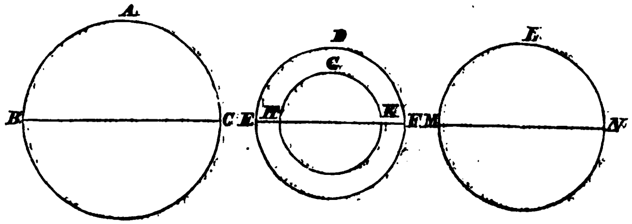
PROPOSITION XVIII.

THEOREM.—*Spheres* have to one another, the triplicate ratio of that which their diameters have.

DEMONSTRATION. Let ABC, DEF be two spheres, of which the diameters are BC, EF: the sphere ABC shall have to the sphere DEF, the triplicate ratio of that which BC has to EF.

For if it has not, the sphere ABC must have to a sphere either less or greater than DEF, the triplicate ratio of that which BC has to EF. First, let it have that ratio to a less, viz. to the sphere GHK; and let the sphere DEF have the same center with GHK: *and in the greater sphere* DEF *inscribe a solid polyhedron, the su-*

perficies of which does not meet the lesser sphere GHK (*a*); *and in the sphere* ABC *inscribe another similar to that in the sphere* DEF:



(*a*) XII. 17. (*b*) XII. 17, Cor. (*c*) V. 14.

therefore the solid polyhedron in the sphere ABC has to the solid polyhedron in the sphere DEF, the triplicate ratio of that which BC has to EF (*b*). But the sphere ABC has to the sphere GHK, the triplicate ratio of that which BC has to EF; therefore, as the sphere ABC is to the sphere GHK, so is the solid polyhedron in the sphere ABC to the solid polyhedron in the sphere DEF: but the sphere ABC is greater than the solid polyhedron in it; therefore also the sphere GHK is greater than the solid polyhedron in the sphere DEF (*c*): but it is also less, because it is contained within it, which is impossible: therefore the sphere ABC has not to any sphere less than DEF, the triplicate ratio of that which BC has to EF. In the same manner it may be demonstrated that the sphere DEF has not to any sphere less than ABC, the triplicate ratio of that which EF has to BC. Nor can the sphere ABC have to any sphere greater than DEF, the triplicate ratio of that which BC has to EF: for if it can, let it have that ratio to a greater sphere LMN: therefore by inversion, the sphere LMN has to the sphere ABC, the triplicate ratio of that which the diameter EF has to the diameter BC. But as the sphere LMN is to ABC, so is the sphere DEF to some sphere which must be less than the sphere ABC (*c*), because the sphere LMN is greater than the sphere DEF; therefore the sphere DEF has to a sphere less than ABC, the triplicate ratio of that which EF has to BC; which was shown to be impossible: therefore the sphere ABC has not to any sphere greater than DEF, the triplicate ratio of that which BC has to EF: and it was demonstrated that neither has it that ratio to any sphere less than DEF. Therefore, *the sphere* ABC *has to the sphere* DEF, *the triplicate ratio of that which* BC *has to* EF.

A CLASSIFIED INDEX

TO THE

FOURTH, FIFTH, SIXTH, ELEVENTH, AND TWELFTH BOOKS

OF THE

ELEMENTS OF EUCLID.

THEOREMS.

C. *Comparison of Triangles as to Equality.*

	Hypotheses.	Consequences.
VI. 4.	If triangles are equiangular .	The sides about the equal angles are proportionals. The sides which are opposite to the equal angles are homologous.
VI. 6.	If two triangles have one angle in each equal, and the sides about the equal angles proportional.	The triangles are equiangular, And have those angles equal which the equal sides subtend.
VI. 5.	If two triangles have their sides proportional.	They are equiangular, And the equal angles are subtended by the homologous sides.
VI. 32. . . .	If two triangles have two sides in the one proportional to two sides in the other, And be joined at one angle so as to have their homologous sides parallel to one another.	The remaining sides shall be in a straight line.
VI. 7.	Or have the angles opposite to one pair of the homologous sides equal; and those opposite to the other pair, either both less, or both not less than a right angle.	The triangles are equiangular, And the angles contained by the proportional sides are equal.

	HYPOTHESES.	CONSEQUENCES.
VI. 23, cor. 1.	If triangles have an angle of the one equal to an angle of the other.	They are to one another as the rectangles under the sides about those angles.
VI. 23, cor. 2.	If triangles are equiangular.	They are to one another as the rectangles under their bases and altitudes.
VI. 19.	If triangles are similar	They are to one another in the duplicate ratio of their homologous sides.
VI. 15.	If equal triangles have an angle of the one equal to an angle of the other.	Their sides about the equal angles are reciprocally proportional.
VI. 15.	If triangles have an angle in the one equal to an angle in the other, and their sides about the equal angles reciprocally proportional.	They are equal to one another.
VI. 19, cor.	If three straight lines be proportionals.	As the first is to the third, so is any triangle upon the first, to a similar and similarly-described triangle upon the second.

D. *On the Relations between the Sides and Angles of Triangles.*

	HYPOTHESES.	CONSEQUENCES.
VI. 31.	If a triangle be right-angled.	The rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly-described figures upon the sides containing the right angle.

E. *On the Relations of Lines drawn in Triangles.*

	HYPOTHESES.	CONSEQUENCES.
VI. 2.	If a straight line be parallel to the base of a triangle.	It cuts the other sides, or those sides produced, so that their segments between the parallel and the base have the same ratio to their segments between the parallel and the vertex.

	HYPOTHESES.	CONSEQUENCES.
VI. 2, cor. . .	If several parallels be drawn to the base of a triangle.	Every pair of corresponding segments in each side will be proportional.
VI. B. (p. 106).	If an angle of a triangle be bisected by a straight line, which likewise cuts the base.	The rectangle under the sides of the triangle is equal to the rectangle under the segments of the base, together with the square on the straight line which bisects the angle.
VI. 3.	Idem	The segments of the base shall have the same ratio which the other sides of the triangle have to one another.
VI. 3, cor. . .	If a straight line bisects both the angle and base of a triangle.	The triangle is isosceles.
VI. C. (p. 106).	If from any angle of a triangle a straight line be drawn perpendicular to the base.	The rectangle under the sides of the triangle is equal to the rectangle under the perpendicular and the diameter of the circle described about the triangle.
VI. 8.	If in a right-angled triangle, a perpendicular be drawn from the right angle to the base.	The triangles on each side of it are similar to the whole triangle, and to one another.
VI. 8, cor. . .	Idem	The perpendicular is a mean proportional between the segments of the base. Each of the sides is a mean proportional between the base and its segment adjacent to that side.
VI. 3.	If a straight line drawn from any angle of a triangle divide the opposite side into segments which have the same ratio as the adjacent sides.	The straight line bisects the angle.
VI. 2. . . .	In a triangle, if the sides, or sides produced, be cut by a straight line, so that their segments between the straight line and the base have the same ratio as their segments between the straight line and the vertex.	The straight line is parallel to the base.
VI. A. (p. 76).	If an exterior angle of a triangle be bisected by a straight line, which also cuts the base produced.	The segments between the bisecting line and the extremities of the base, have the same ratio to one another as the other sides of the triangle have.

	HYPOTHESES.	CONSEQUENCES.
VI. A. (p. 76).	If a straight line drawn from the vertex of a triangle cuts the base produced, so that its segments have the same ratio which the other sides of the triangle have.	The straight line bisects the exterior angle of the triangle.
VI. A, cor. (p. 76). . .	If both the exterior angle and the adjacent interior angle of a triangle be bisected by straight lines which cut the base and its production.	The base thus produced is harmonically divided.

F. *Comparison of Parallelograms with Triangles.*

	HYPOTHESES.	CONSEQUENCES.
VI. 1.	If triangles and parallelograms have the same altitude.	They are to one another as their bases.
VI. 1, cor. 1. .	If triangles and parallelograms have equal altitudes.	
VI. 1, cor. 2. .	If triangles and parallelograms have equal bases.	They are to one another as their altitudes.
VI. 1, cor. 3. . .	If neither the bases nor altitudes of triangles and parallelograms are equal.	They are to one another in the compound ratio of their bases and altitudes.

G. *Comparison of Parallelograms as to Equality.*

	HYPOTHESES.	CONSEQUENCES.
VI. 23, cor. 2. .	If parallelograms are equiangular.	They are to one another as the rectangles under their bases and altitudes.
VI. 23. . . .	Idem	They have to one another the ratio which is compounded of the ratios of their sides.
VI. 14. . . .	If equal parallelograms have an angle of the one equal to an angle of the other.	Their sides about the equal angles are reciprocally proportional.
VI. 14. . . .	If parallelograms have an angle of the one equal to an angle of the other, and their sides about the equal angles reciprocally proportional.	They are equal to one another.
VI. 24. . . .	If parallelograms are about the diameter of any parallelogram.	They are similar to the whole and to one another.
VI. 26. . . .	If two similar parallelograms have a common angle, and be similarly situated.	They are about the same diameter.

I. *Comparison of Rectangles contained by Straight Lines and their Segments.*

	HYPOTHESES.	CONSEQUENCES.
VI. 17. . . .	If three straight lines be proportionals.	The rectangle under the extremes is equal in area to the square on the mean.
VI. 16. . . .	If four straight lines be proportionals.	The rectangle under the extremes is equal in area to the rectangle under the means.
VI. 17. . . .	If in three straight lines the rectangle under the extremes is equal in area to the square on the mean.	The lines are proportionals.
VI. 16. . . .	If in four straight lines the rectangle under the extremes be equal in area to the rectangle under the means.	The lines are proportionals.

K. *Of Polygons.*

	HYPOTHESES.	CONSEQUENCES.
VI. 20. . . .	If polygons are similar . .	They may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have. The polygons have to one another the duplicate ratio of that which their homologous sides have.
VI. 20, cor. 3. .	Idem	Their perimeters are as their homologous sides.
VI. 21. . . .	If rectilineal figures are similar to the same rectilineal figure.	They are similar to one another.
VI. 20, cor. 2. .	If three straight lines be proportionals.	As the first is to the third, so is any rectilineal figure upon the first to a similar and similarly-described rectilineal figure upon the second.
VI. 22. . . .	If four straight lines be proportionals.	The similar rectilineal figures similarly-described upon them shall also be proportionals.
VI. 22. . . .	If the similar rectilineal figures similarly-described upon four straight lines be proportionals.	The lines shall also be proportionals.

L. *Relative to Circles generally.*

	Hypotheses.	Consequences.
XII. 2. . . .	If figures are circles . . .	They are to one another as the squares on their diameters.

P. *On the Angles in a Circle.*

	Hypotheses.	Consequences.
VI. 33. . . .	If angles are in equal circles.	Whether at the centers or circumferences, they have the same ratio which the circumferences on which they stand have to one another.

S. *Relative to Inscribed Figures.*

	Hypotheses.	Consequences.
VI. 27. . . .	If a parallelogram be constructed on the half of one of the sides of a triangle in which it is inscribed.	It is the greatest parallelogram which can be inscribed in the triangle.
IV. 4, cor. 2. .	If a circle is inscribed in a triangle.	The triangle is equal in area to the rectangle under the radius of the circle, and half the sum of the three sides of the triangle.
IV. 4, cor. 1. .	If straight lines bisect the three angles of a triangle.	They meet in the center of the inscribed circle.
IV. 7, cor. . .	If a square is circumscribed about a circle.	It is equal in area to twice the square inscribed in the circle.
VI. D. (p. 107).	If a rectangle be contained under the diagonals of a quadrilateral figure inscribed in a circle.	It is equal to both the rectangles contained by its opposite sides.
XI. Lem. 2. (p. 186). .	If two trapeziums be inscribed in circles; and if two of the sides of each be parallel to each other; and the other four sides be all equal to one another, but the parallel sides of one greater than the parallel sides of the other, each to each.	The straight line drawn from the center to the circumference of the circle in which the greater sides are, is greater than the straight line similarly drawn in the other circle.

	HYPOTHESES.	CONSEQUENCES.
IV. 5 A. . . .	If a rectilineal figure be equilateral and equiangular.	It may have one circle circumscribed about it, and another inscribed in it. And the same point is the center of both circles.
XII. 1. . . .	If similar polygons are inscribed in circles.	They are to one another as the squares on their diameters.
IV. 11, cor. . .	If an equiangular figure is inscribed in a circle.	It is equilateral.
IV. 11, cor. . .	If an equilateral figure is inscribed in a circle.	It is equiangular.
IV. 5 B. . . .	If any equilateral and equiangular rectilineal figure be inscribed in a circle.	Tangents to the circle drawn through the angular points, will form an equilateral and equiangular figure of the same number of sides, circumscribed about the circle.
IV. 15, cor. . .	If a hexagon be inscribed in a circle.	The radius of the circle is equal to the side of the hexagon.

T. *Of the Multiples of Magnitudes.*

	HYPOTHESES.	CONSEQUENCES.
XII. Lem. 1. (p. 160). .	If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half, and so on.	There shall at length remain a magnitude less than the least of the proposed magnitudes.
V. 5.	If one magnitude be the same multiple of another, which a part taken from the first is of a part taken from the other.	The first remainder is the same multiple of the second that the first magnitude is of the second.
V. 6.	If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the two first.	The remainders are either equal to these others, or equimultiples of them.
V. 1.	If any number of magnitudes be equimultiples of as many others, each of each.	What multiple soever any one of the first is of its part, the same multiple shall all the first magnitudes taken together, be of all the others taken together.
V. a (p. 84). .	If the first be the same multiple or submultiple of the second that the third is of the fourth.	The first is to the second as the third is to the fourth.

	Hypotheses.	Consequences.
V. D. (p. 36).	If the first be to the second as the third to the fourth, and if the first be a multiple or submultiple of the second.	The third is the same multiple or submultiple of the fourth.
V. 3.	If the first be the same multiple of the second which the third is of the fourth, and if of the first and third there be taken equimultiples.	These shall be equimultiples, the one of the second, and the other of the fourth.
V. 2.	If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth.	Then shall the first, together with the fifth, be the same multiple of the second, that the third, together with the sixth, is of the fourth.

V. *Of the Ratios of Magnitudes.*

	Hypotheses.	Consequences.
V. B. (p. 33).	If four magnitudes are proportionals.	They are proportionals also when taken inversely.
V. E. (p. 53).	Idem	They are also proportionals by conversion.
V. 16.	If four magnitudes of the same kind be proportionals.	They are also proportionals when taken alternately.
V. 25.	Idem	The greatest and least of them together are greater than the other two together.
V. 4.	If the first of four magnitudes has the same ratio to the second which the third has to the fourth.	Then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth.
V. 4, cor.	Idem	Any equimultiples whatever of the first and third shall have the same ratio to the second and fourth; and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.
V. 18.	Idem	The first and second together shall be to the second as the third and fourth together to the fourth.
V. 14.	And if the first be greater than the third.	The second shall be greater than the fourth; and if equal, equal; and if less, less.

	HYPOTHESES.	CONSEQUENCES.
V. A. (p. 38).	Or if the first be greater than the second.	The third is also greater than the fourth; and if equal, equal; if less, less.
V. 24.	If the first have to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth.	The first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.
V. 24, cor. 1.	Idem	The difference of the first and fifth shall be to the second as the difference of the third and sixth is to the fourth.
V. 13.	If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth.	The first shall also have to the second a greater ratio than the fifth has to the sixth.
V. 13, cor.	If the first has a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth.	The first has a greater ratio to the second than the fifth has to the sixth.
V. 12.	If any number of magnitudes be proportionals.	As one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents taken together.
V. 17.	If magnitudes, taken jointly, be proportionals.	They shall also be proportionals when taken separately.
V. 19.	If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other.	The remainder shall be to the remainder as the whole is to the whole.
V. 19, cor.	Idem	The remainder shall be to the remainder as the magnitude taken from the first is to that taken from the other.
V. 15.	If magnitudes have a ratio to one another.	Their equimultiples have the same ratio.
V. 9.	If magnitudes have the same ratio to the same magnitude.	They are equal to one another.

	HYPOTHESES.	CONSEQUENCES.
V. 20.	If there be three magnitudes, and other three, which, taken two and two, have the same ratio, then if the first be greater than the third.	The fourth shall be greater than the sixth; and if equal, equal; and if less, less.
V. 21.	If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order, then, if the first magnitude be greater than the third.	The fourth shall be greater than the sixth; and if equal, equal; and if less, less.
V. 22.	If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio.	The first has to the last of the first magnitudes the same ratio which the first has to the last of the others.
V. 23.	If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio.	The first has to the last of the first magnitudes the same ratio which the first has to the last of the others.
V. 7.	If magnitudes are equal . .	They have the same ratio to the same magnitude.
V. 8.	If two magnitudes are unequal.	The greater has a greater ratio to any other magnitude than the less has.
V. 10.	If a magnitude has a greater ratio than another has to the same magnitude.	It is the greater of the two.
V. 11.	If ratios are equal to the same ratio.	They are equal to one another.
V. F. (p. 63). .	If ratios are compounded of the same ratios.	They are the same with one another.
V. G. (p. 64). .	If several ratios be the same with several ratios, each to each.	The ratio which is compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.
V. H. (p. 65). .	If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first, be the same with one of the last ratios, or with the ratio compounded of any of the last.	Then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.

	HYPOTHESES.	CONSEQUENCES.
V. K. (p. 67).	If there be any number of ratios, and any number of other ratios such that the ratio compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same, each to each, with the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same with several of the first ratios, each to each, be the same with one of the last ratios, or with the ratio compounded of ratios which are the same, each to each, with several of the last ratios.	Then the ratio compounded of ratios which are the same with the remaining ratios of the first, each to each, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of ratios which are the same with those remaining of the last, each to each, or with the remaining ratio of the last.

W. *Of the Relations of Lines to Planes.*

	HYPOTHESES.	CONSEQUENCES.
XI. 1.	If one part of a straight line is above a plane.	Another part cannot be in it.
XI. 18.	If a straight line be at right angles to a plane.	Every plane which passes through it shall be at right angles to that plane.
XI. 4.	If a straight line stand at right angles to each of two straight lines, in the point of their intersection.	It shall also be at right angles to the plane which passes through them.
XI. 13.	If two straight lines be drawn from a given point either in or above a plane.	They cannot both be at right angles to it.
XI. 2.	If two straight lines cut one another.	They are in one plane.
XI. 6.	If two straight lines be at right angles to the same plane.	They shall be parallel to one another.
XI. 7.	If two straight lines be parallel.	The straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.
XI. 8.	And one of them is at right angles to a plane.	The other shall also be at righ angles to the same plane.

	HYPOTHESES.	CONSEQUENCES.
XI. 9.	If two straight lines are each of them parallel to the same straight line, and not in the same plane with it.	They are parallel to one another.
XI. 10. . .	If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two.	The first two and the other two shall contain equal angles.
XI. 2.	If three straight lines meet one another.	They are in one plane.
XI. 5.	If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point.	These three straight lines are in one and the same plane.
XI. 35. . . .	If from the vertices of two equal plane angles, there be drawn two straight lines, elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the first-named angles are; and from the points in which they meet the planes straight lines be drawn to the vertices of the angles first-named.	These straight lines shall contain equal angles with the straight lines which are above the planes of the angles.
XI. 35, cor. . .	If from the vertices of two equal plane angles, there be elevated two equal straight lines, containing equal angles with the sides of the angles, each to each.	The perpendiculars drawn from the extremities of the equal straight lines to the planes of the first angles, are equal to one another.

X. *Of the Relations of Planes to one another.*

	HYPOTHESES.	CONSEQUENCES.
XI. 3.	If two planes cut one another.	Their common section is a straight line.
XI. 19. . . .	If two planes which cut one another be each of them perpendicular to a third plane.	Their common section shall be perpendicular to the same plane.

	HYPOTHESES.	CONSEQUENCES.
XI. 14. . . .	If the same straight line is perpendicular to each of two planes.	They are parallel to one another.
XI. 17. . . .	If two straight lines be cut by parallel planes.	They shall be cut in the same ratio.
XI. 15. . . .	If two straight lines meeting one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two.	The plane which passes through these is parallel to the plane passing through the others.
XI. 38. . . .	If a plane be perpendicular to another plane, and a straight line be drawn from a point in one of the planes perpendicular to the other plane.	This straight line shall fall on the common section of the planes.
XI. 16. . . .	If two parallel planes be cut by another plane.	Their common sections with it are parallels.
XI. 22. . . .	If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal.	A triangle may be made of the straight lines that join the extremities of those equal straight lines.

V. *Of Solid Angles.*

	HYPOTHESES.	CONSEQUENCES.
XI. 21. . . .	If an angle is a solid angle .	It is contained by plane angles which together are less than four right angles.
XI. 20. . . .	If a solid angle be contained by three plane angles.	Any two of them are greater than the third.
XI. A. (p. 134) .	If each of two solid angles be contained by three plane angles, which are equal to one another, each to each.	The planes in which the equal angles are, have the same inclination to one another.
XI. B. (p. 135).	If two solid angles be contained, each by three plane angles, which are equal to one another, each to each, and alike situated.	These solid angles are equal to one another.

Z. *Of Solid Figures.*

	HYPOTHESES.	CONSEQUENCES.
XI. 36.	If three straight lines be proportionals.	The solid parallelopiped described from all three, as its sides, is equal to the equilateral parallelopiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the figure.
XI. 37.	If four straight lines be proportionals.	The similar solid parallelopipeds similarly described from them shall also be proportionals.
XI. 33, cor.	If four straight lines be continual proportionals.	As the first is to the fourth, so is the solid parallelopiped described from the first to the similar solid similarly described from the second.
XI. 33.	If solid parallelopipeds are similar.	They are to one another in the triplicate ratio of their homologous sides.
XI. 32.	If solid parallelopipeds have the same altitude;	They are to one another as their bases.
XI. 31.	And are upon equal bases	They are equal to one another.
XI. 7, cor. 2.	If prisms are of equal altitudes;	They are to one another as their bases.
XI. 32, cor.	And are upon triangular bases	Idem.
XI. 34.	If solid parallelopipeds are equal.	Their bases and altitudes are reciprocally proportional.
XI. 40.	If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram, and the base of the other a triangle: if the parallelogram be double of the triangle.	The prisms shall be equal to one another.
XI. 24.	If a solid be contained by six planes, two and two of which are parallel.	The opposite planes are similar and equal parallelograms.
XI. c. (136).	If solid figures are contained by the same number of equal and similar planes alike situated, and having none of their solid angles contained by more than three plane angles.	They are equal and similar to one another.

	Hypotheses.	Consequences.
XI. D. (p. 148).	If solid parallelopipeds are contained by parallelograms equiangular to one another, each to each.	They have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.
XI. 29 and 30.	If solid parallelopipeds are upon the same base, and of the same altitude, whether their insisting straight lines are terminated in the same straight lines in the plane opposite to the base, or not.	They are equal to one another.
XI. 25.	If a solid parallelopiped be cut by a plane parallel to two of its opposite planes.	It divides the whole into two solids, the base of one of which shall be to the base of the other, as the one solid is to the other.
XI. 28.	If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes.	It shall be cut into two equal parts.
XI. 39.	In a solid parallelopiped, if the sides of two of the opposite planes be divided, each into two equal parts.	The common section of the planes passing through the points of division, and the diameter of the solid parallelopiped, cut each other into two equal parts.
XII. 7.	If a prism has a triangular base.	It may be divided into three pyramids that have triangular bases, and are equal to one another.
XII. 3.	If a pyramid has a triangular base.	It may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than half of the whole pyramid.
XII. 8.	If pyramids are similar, and have triangular bases;	They are to one another in the triplicate ratio of that of their homologous sides.
XII. 8, cor.	Or have multangular bases.	
XII. 5.	If pyramids of the same altitude have triangular bases;	They are to one another as their bases.
XII. 6.	Or have polygonal bases.	
XII. 9.	If triangular pyramids are equal.	Their bases and altitudes are reciprocally proportional.
XII. 9.	If the bases and altitudes of triangular pyramids are reciprocally proportional.	They are equal to one another.
XII. 7, cor. 1.	If a solid is a pyramid.	It is the third part of a prism which has the same base and altitude.

	HYPOTHESES.	CONSEQUENCES.
XII. 4. . . .	If there be two pyramids of the same altitude upon triangular bases, and each of them be divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on.	As the base of one of the first two pyramids is to the base of the other, so shall all the prisms in one of them be to all the prisms in the other, that are produced by the same number of divisions.
XII. 12. . . .	If cones and cylinders are similar.	They have to one another the triplicate ratio of that which the diameters of their bases have.
XII. 11. . . .	If cones and cylinders are of the same altitude.	They are to one another as their bases.
XII. 15. . . .	If cones and cylinders are equal.	Their bases and altitudes are reciprocally proportional.
XII. 15. . . .	If the bases and altitudes of cones and cylinders be reciprocally proportional.	They are equal to one another.
XII. 14. . . .	If cones and cylinders are upon equal bases.	They are to one another as their altitudes.
XII. 10. . . .	If a solid is a cone	It is the third part of a cylinder which has the same base and altitude.
XII. 13. . . .	If a cylinder be cut by a plane parallel to its opposite planes, or bases.	It divides the cylinder into two cylinders, one of which is to the other as the axis of the first is to the axis of the other.
XII. 18. . . .	If solids are spheres . . .	They have to one another the triplicate ratio of that which their diameters have.
XII. 17, cor. .	If in the lesser of two concentric spheres there be inscribed a solid polyhedron, by drawing straight lines betwixt the points in which the straight lines from the center of the spheres, drawn to all the angles of the solid polyhedron in the greater sphere, meet the superficies of the lesser, in the same order in which are joined the points in which the same lines from the center meet the superficies of the greater sphere.	The solid polyhedrons shall have to one another the triplicate ratio of the diameters of their circumscribing spheres.

PROBLEMS.

A. *Relating to Straight Lines.*

XI. 11.	To draw a straight line perpendicular to a plane, from a given *point* above it.
XI. 13.	To draw a straight line at right angles to a given *plane*, from a *point* given in that plane.
VI. 9.	From a given *finite straight line* to cut off any required part.
VI. 10.	To divide a given *straight line* similarly to a given *divided straight line.*
VI. 30.	To cut a given *straight line* in extreme and mean ratio.
VI. 28.	To divide a given *straight line* into two parts, such that parallelograms of equal altitude may be constructed upon them, one equal to a given *rectilineal figure*, and the other similar to a given *parallelogram ;* the rectilineal figure not being greater than the parallelogram constructed on half the given line, and similar to the given parallelogram.
VI. 13.	To find a mean proportional between *two* given *straight lines.*
VI. 11.	To find a third proportional to *two* given *straight lines.*
VI. 12.	To find a fourth proportional to *three* given *straight lines.*
VI. 29.	To produce a given *straight line* so that a parallelogram similar to a given *one* being constructed on the produced part, another parallelogram of equal altitude constructed on the whole line produced, may be equal to a given *rectilineal figure.*

B. *Relating to Rectilineal Angles.*

IV. 10, cor. 3.	To divide a given *right angle* into five equal parts.

C. *Relating to Triangles.*

IV. 10.	To construct an isosceles triangle, in which each of the angles at the base shall be double of the angle opposite to the same.

F. *Relating to Inscribed Figures.*

IV. 1.	In a given *circle* to inscribe a straight line equal to a given *straight line*, which is not greater than the diameter of the circle.
IV. 4.	To inscribe a circle in a given *triangle.*
IV. 5.	To circumscribe a circle about a given *triangle.*
IV. 2.	In a given *circle* to inscribe a triangle equiangular to a given *triangle.*

IV. 3.	About a given *circle* to circumscribe a triangle equiangular to a given *triangle*.
IV. 8.	To inscribe a circle in a given *square*.
IV. 6.	To inscribe a square in a given *circle*.
IV. 9.	To circumscribe a circle about a given *square*.
IV. 7.	To circumscribe a square about a given *circle*.
IV. 13.	To inscribe a circle in a given *equilateral and equiangular pentagon*.
IV. 14.	To circumscribe a circle about a given *equilateral and equiangular pentagon*.
IV. 12.	To circumscribe an equilateral and equiangular pentagon about a given *circle*.
IV. 11.	To inscribe an equilateral and equiangular pentagon in a given *circle*.
IV. 15.	To inscribe an equilateral and equiangular hexagon in a given *circle*.
IV. 16.	To inscribe an equilateral and equiangular quindecagon in a given *circle*.
XII. 16.	In the greater of two given *circles that have the same center*, to inscribe a polygon of an even number of equal sides, that shall not meet the lesser circle.

G. *Relating to Polygons.*

VI. 18.	On a given *straight line* to construct a rectilineal figure similar, and similarly situated to a given *rectilineal figure*.
VI. 25.	To construct a rectilineal figure which shall be similar to *one*, and equal to *another* given *rectilineal figure*.

H. *Relating to Solid Angles.*

XI. 23.	To make a solid angle which shall be contained by *three* given *plane angles*, any two of them being greater than the third, and all three together less than four right angles.
XI. 26.	At a given *point* in a given *straight line* to make a solid angle equal to a given *solid angle contained by three plane angles*.

I. *Relating to Solid Figures.*

XI. 27.	To describe, from a given *straight line*, a solid parallelopiped similar and similarly situated to *one* given.
XII. 17.	In the greater of two given *spheres which have the same center*, to inscribe a solid polyhedron, the superficies of which shall not meet the lesser sphere.

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BY HENRY LAW,

CIVIL ENGINEER.

	HYPOTHESES.	CONSEQUENCES.
V. D. (p. 86).	If the first be to the second as the third to the fourth, and if the first be a multiple or submultiple of the second.	The third is the same multiple or submultiple of the fourth.
V. 3.	If the first be the same multiple of the second which the third is of the fourth, and if of the first and third there be taken equimultiples.	These shall be equimultiples, the one of the second, and the other of the fourth.
V. 2.	If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth.	Then shall the first, together with the fifth, be the same multiple of the second, that the third, together with the sixth, is of the fourth.

V. *Of the Ratios of Magnitudes.*

	HYPOTHESES.	CONSEQUENCES.
V. B. (p. 33).	If four magnitudes are proportionals.	They are proportionals also when taken inversely.
V. E. (p. 53).	Idem	They are also proportionals by conversion.
V. 16.	If four magnitudes of the same kind be proportionals.	They are also proportionals when taken alternately.
V. 25.	Idem	The greatest and least of them together are greater than the other two together.
V. 4.	If the first of four magnitudes has the same ratio to the second which the third has to the fourth.	Then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth.
V. 4, cor.	Idem	Any equimultiples whatever of the first and third shall have the same ratio to the second and fourth; and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.
V. 18.	Idem	The first and second together shall be to the second as the third and fourth together to the fourth.
V. 14.	And if the first be greater than the third.	The second shall be greater than the fourth; and if equal, equal; and if less, less.

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	HYPOTHESES.	CONSEQUENCES.
V. 20.	If there be three magnitudes, and other three, which, taken two and two, have the same ratio, then if the first be greater than the third.	The fourth shall be greater than the sixth; and if equal, equal; and if less, less.
V. 21.	If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order, then, if the first magnitude be greater than the third.	The fourth shall be greater than the sixth; and if equal, equal; and if less, less.
V. 22.	If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio.	The first has to the last of the first magnitudes the same ratio which the first has to the last of the others.
V. 23.	If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio.	The first has to the last of the first magnitudes the same ratio which the first has to the last of the others.
V. 7.	If magnitudes are equal . .	They have the same ratio to the same magnitude.
V. 8.	If two magnitudes are unequal.	The greater has a greater ratio to any other magnitude than the less has.
V. 10.	If a magnitude has a greater ratio than another has to the same magnitude.	It is the greater of the two.
V. 11.	If ratios are equal to the same ratio.	They are equal to one another.
V. F. (p. 63). .	If ratios are compounded of the same ratios.	They are the same with one another.
V. G. (p. 64). .	If several ratios be the same with several ratios, each to each.	The ratio which is compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.
V. H. (p. 65). .	If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first, be the same with one of the last ratios, or with the ratio compounded of any of the last.	Then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.